CHAPTER 4
Working with algebraic expressions

In this chapter you will

- Add, subtract, multiply and divide like and unlike terms
- Use the rules for the order of operations to simplify expressions
- Find the product of a monomial and a binomial
- Find the product of two binomials
- Square a binomial
- Find the product of a binomial and a trinomial
- Factorise by taking out common factors
- Factorise by taking out binomials as common factors
- Group to find a common factor
- Factorise the difference of two squares
- Factorise trinomials with two positive signs
- Factorise trinomials with a negative and a positive sign
- Factorise trinomials when the last sign is negative
- Factorise more trinomials
- Divide by a monomial

This chapter covers material from Topic 2: Functions
<u>SUBJECT OUTCOME 2.2:</u>
Manipulate algebraic expressions
<u>Learning Outcome 1:</u> Find products of binomials with trinomials
<u>Learning Outcome 2:</u> Factorise by grouping and factorising trinomials
<u>Learning Outcome 3:</u> Simplify fractions with monomial denominators

§ 4.1 ADDITION, SUBTRACTION, MULTIPLICATION AND DIVISION OF LIKE AND UNLIKE TERMS

- If we add one flower to one flower we have two flowers. ❀ + ❀= 2❀

  In the same way, 3❀ + 2❀ = 5❀

  We can add a number of flowers to a number of flowers because they are **like** (the same) and the answer will be a number of flowers. However, if we add one apple to one flower, the answer is neither two apples nor two flowers, since these are **unlike** (not the same). We **cannot** say

  🚲 + ❀ = 2.

- Similarly, in algebra, we can add or subtract **like terms**.
  Example 1: $x + x = 2x$
  Example 2: $3x + 2x = 5x$. We have added like terms.

- We cannot find a simpler answer to $x + y$ since these are **unlike terms**.
  Example Are $3x^2$ and $2x^3$ like terms?
  Remember that $x^2 = x \times x$ whereas $x^3 = x \times x \times x$, so these are **unlike** terms.
  So $3x^2 + 2x^3 = 3x^2 + 2x^3$ since this expression cannot be simplified.

- x and y are **unlike**, but we can multiply them: $x \times y = xy$.

  Note:
  - $2 \times 3 = 3 \times 2$ so $x \times y = y \times x = xy$. Then $xy + yx = 2xy$ or $2yx$
  - xy means $1xy$

EXAMPLE		SOLUTION
1)	Choose the like terms from the following: $7a^3c^2,\ 3a^3c,\ 4a^3c^2,\ -7ac^2,\ -a^3c^2,\ a^3c,\ a^2c^3,\ 3c^2a^3$	The like terms are: $7a^3c^2,\ 4a^3c^2,\ -a^3c^2,\ 3c^2a^3$
2)	Simplify $3a + 4d - 4a + 7d$	$3a + 4d - 4a + 7d$ $= (3a - 4a) + (4d + 7d) = -a + 11d$

Exercise 4.1

Simplify where possible:

1) $3x - 1 + 2x + 5 =$ ……………………..…..………………………………………………………………
2) $4a + 3b - 2a - b =$ ……………………………………………………………………………………..
3) $2x^2 + 4x - 2x + x^2 =$ ……………………………………………………………………………………
4) $3a + 2b + 5c =$ ……………………………………………………………………………………..
5) $3p + 4p^2 + 2p^3 =$ …………………………………………………………………………………
6) $4pq + 6pq - 3pq =$ ……………………………………………………………………………..
7) $3xy - 3x - 2yx + 6xy - xy + 2x =$ ……………………………………………….…………………
8) $3ab^2 - 8ab + 7a^2b + 6ab - 4a^2b =$ …………………………………………………..………...........
9) $3xy - 7xy + 2yx =$ ………………………………………………………………………………………
10) $9x^2 - 9 - x^2 =$ ……….………………….…………………………………………………………………
11) $2x^2y^3 + 4x^3y^2 - x^2y^3 + 5y^2x^3 =$ ……………………………………………………………………

§ 4.2 ORDER OF OPERATIONS

- **BEMDAS** reminds you of the order of the operations. It stands for:
  Brackets and **E**xponents,
  Multiplication and **D**ivision,
  Addition and **S**ubtraction.
  This means that you first do the calculations inside brackets, then work out exponents.
  After that you deal with multiplication and division *in order* from left to right.
  Finally you add and subtract *in order* from left to right..
- **Note:** In algebra we usually write the number first, so $2 \times x$ and $x \times 2$ are both written $2x$

EXAMPLE		SOLUTION
Simplify:		
1)	$3x - 3 - 5x - 1$	$3x - 3 - 5x - 1 = 3x - 5x - 3 - 1 = -2x - 4$
2)	$6x^3 \div 2x^2 + 7x$	$6x^3 \div 2x^2 + 7x = (6x^3 \div 2x^2) + 7x = 3x + 7x = 10x$
3)	$2a \times 3b + 4 \times 5a \times \text{b}$	$2a \times 3b + 4 \times 5a \times \text{b} = (2a \times 3b) + (4 \times 5a \times \text{b}) = 6ab + 20ab = 26ab$
4)	$(3x \times 4)^2 - 4x \times x$	$(3x \times 4)^2 - 4x \times x = (12x)^2 - 4x^2 = 144x^2 - 4x^2 = 140x^2$
5)	$(3p - p) \times (6p + 2p)$	$(3p - p) \times (6p + 2p) = 2p \times 8p = 16p^2$

Exercise 4.2

Simplify:

1) $(3x + 5x) \times (4x - x) =$ ……………………………………………………………………………..

2) $4x \div 2 - 2 \times x =$ …………………………………………………………………………………

3) $9x - 4x + 3x \times 2 =$ ………………………………………………………………………………

4) $12x^2 \div 3x - x =$ ………………………………………………………………………………

5) $7p^2 - 2p \times 3p + p^2 \times 3 =$ ……………………………………………………………………

6) $4x^2 - x \times x \times 7 =$ ………………………………………………………………………………

7) $8a^2 \div 2a - a =$ ………………………………………………………………………………………

8) $(9x - x) \div 2x + 2x =$ ……………………………………………………………………………

9) $(3p \times 2p)^2 - 6p^4 =$ ……………………………………………………………………………..

10) $4b \times 2b - 3b \times 2b =$ ……………………………………………………………………………..

11) $2 \times 76 \times 5 - 5 =$ ………………………………………………………………………………

12) $-5xy \times 8yx + 2xy =$ ……………………………………………………………………………..

13) $10a^2 + 9a \times 10a =$ ……………………………………………………………………………..

14) $25b \times 3b + 2b \times 4b =$ …………………………………………………………………………

§ 4.3 THE PRODUCT OF A MONOMIAL AND A BINOMIAL

- The **product** of two or more numbers is the answer to the **multiplication** of the numbers. For example, the product of 3 and 4 is 12, the product of 2 and x is $2x$.

- A **monomial** has ONE term.
“mono” means one.

A **mono**cycle has only one wheel

The following are examples of monomials: $2x^2$; $-5a$; 275; $a^2b^3c^2d$

- A **binomial** has TWO terms.
“bi” means two.

A **bi**cycle has two wheels

The following are examples of binomials: $(a + b)$; $(2x^2 - 5)$; $a^2b^3c - 27$

- Remember to multiply the signs (+ or –) using the following rules:

Rules for Signs	Numeric example	Algebraic example
$(+) \times (+) = (+)$	$2 \times 3 = 6$	$a \times b = ab$
$(+) \times (-) = (-)$	$2 \times -3 = -6$	$a \times -b = -ab$
$(-) \times (+) = (-)$	$-2 \times 3 = -6$	$-a \times b = -ab$
$(-) \times (-) = (+)$	$-2 \times -3 = 6$	$-a \times -b = ab$

- $\boldsymbol{a(b + c) = ab + ac}$ is called the distributive law, since ***a* is distributed over both *b* and *c*.**

$$\overset{\times}{}$$
$$\boldsymbol{a\,(b + c) = ab + ac}$$
$$\times$$

- The distributive law also works for multiplication over subtraction: $\boldsymbol{a(b - c) = ab - ac}$

EXAMPLE		SOLUTION
Simplify		
1)	$3(2x - 4)$	$3(2x - 4) = 3.2x - 3.4 = 6x - 12$
2)	$2c(3a + 2b)$	$2c(3a + 2b) = 2c.3a + 2c.2b = 6ac + 4bc$
3)	$(6a - 3)4a$	$(6a - 3)4a = 4a(6a - 3) = 4a.6a - 4a.3 = 24a^2 - 12a$

Exercise 4.3

Simplify the following:

1) $4(2a - 1) =$ ……………………………………………
2) $3(5x - 2y) =$ ……………………………………..
3) $2x(3x + 5) =$ ……………………………………
4) $(2x + 4).3 =$ ………………………………
5) $4x(x - 2) =$ …………………………………….
6) $3p(2p^2 + p) =$ ………………………………
7) $3a(2a^2 - 4b^3) =$ ……………………………..
8) $(9b - 4c)3a =$ ………………………………
9) $5a^2(3a^3 + 4b^3) =$ ……………………………
10) $-3p(2q - 3p + 4r) =$ ………………………

§ 4.4 THE PRODUCT OF TWO BINOMIALS

✧ When we multiply two binomials, we use **FOIL** to remind us what to do. This means that we multiply:

The **FIRSTS:** $(a+b)(c+d)$
$\uparrow \qquad \uparrow$

The **OUTERS:** $(a+b)(c+d)$
$\uparrow \qquad\quad \uparrow$

The **INNERS:** $(a+b)(c+d)$
$\uparrow \quad \uparrow$

The **LASTS:** $(a+b)(c+d)$
$\uparrow \qquad\quad \uparrow$

And then simplify.

Example 1:
Simplify $(x + 2)(x + 3)$

F: first × first $= x \times x = x^2$
O: outer × outer $= x \times -3 = -3x$
I: inner × inner $= 2 \times x = 2x$
L: last × last $= 2 \times -3 = -6$

So $(x + 2)(x + 3)$
$= \boldsymbol{x^2 - 3x + 2x - 6 = x^2 - x - 6}$

Example 2:
Simplify $(x + y)(p + q)$

F: first × first $= \boldsymbol{x \times p = xp}$
O: outer × outer $= \boldsymbol{x \times q = xq}$
I: inner × inner $= \boldsymbol{y \times p = yp}$
L: last × last $= \boldsymbol{y \times q = yq}$

So $(x + y)(p + q) = xp + xq + yp + yq$

EXAMPLE		SOLUTION
Simplify		
1)	$(x + 1)(x - 2)$	$(x + 1)(x - 2) = x^2 - 2x + 1x - 2 = x^2 - x - 2$
2)	$(b - 3)(b - 4)$	$(b - 3)(b - 4) = b^2 - 4b - 3b + 12 = b^2 - 7b + 12$

Exercise 4.4

Remove the brackets and simplify:

1) $(x + 1)(x + 2) =$ ……………………………………………………………………………..……..................

2) $(a - 2)(a + 3) =$ …………………………………………………………………………………………

3) $(x - 1)(x + 5) =$ …………………………………………………………………………………………

4) $(x - 2)(x - 5) =$ …………………………………………………………………………………………

5) $(p + 4)(p - 3) =$ ………………………………………………………………………..……………..

6) $(2a + 3b)(4a - b) =$ …………………………………………………………………………………..

7) $(ab - 2)(ab + 2) =$ ……………………………………………………………………………………

8) $(x - 3)(x + 1) =$ …………………………………………………………………………………………

9) $(6a + 5b)(2a - b) =$ …………………………………………………………………………………..

10) $(5x - 3y)(5x + 3y) =$ ………………………………………………………………………………

§ 4.5 SQUARING A BINOMIAL

- Remember FOIL means First, Outer, Inner, Last.
- Using FOIL, $\mathbf{(a + b)^2 = (a + b)(a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2}$

EXAMPLE		SOLUTION
Simplify		
1)	$(x + 3)^2$	$(x + 3)^2 = (x + 3)(x + 3) = x^2 + 6x + 9$
2)	$(x - 4)^2$	$(x - 4)^2 = (x - 4)(x - 4) = x^2 - 8x + 16$
3)	$(2x - 3)^2$	$(2x - 3)^2 = (2x - 3)(2x - 3) = 4x^2 - 12x + 9$
4)	$(x - 3y)^2$	$(x - 3y)^2 = (x - 3y)(x - 3y) = x^2 - 6xy + 9y^2$
5)	$(2a + 3b)^2$	$(2a + 3b)^2 = (2a + 3b)(2a + 3b) = 4a^2 + 12ab + 9b^2$
6)	$2(x + 1)^2$	$2(x + 1)(x + 1) = 2(x^2 + x + x + 1) = 2(x^2 + 2x + 1) = 2x^2 + 4x + 2$

<u>Exercise 4.5</u>

Remove the brackets and write down the answer without any working:

1) $(y + 5)^2$ = ……………………………………………………………………………………………..
………………………………………………………………………………………………………
2) $(p - 4)^2$ = ………………………………………………………………………………………
………………………………………………………………………………………………………
3) $(y - 3)^2$ = ………………………………………………………………………………………
………………………………………………………………………………………………………
4) $(5b + 6)^2$ = ……………………………………………………………………………………..
………………………………………………………………………………………………………
5) $(3a - 2b)^2$ = ……………………………………………………………………………………
………………………………………………………………………………………………………
6) $(a^2 - 3a)^2$ = ……………………………………………………………………………………
………………………………………………………………………………………………………
7) $(x^2 - y)^2$ = ………………………………………………………………………………………
………………………………………………………………………………………………………
8) $5(x + 3)^2$ = ……………………………………………………………………………………..
………………………………………………………………………………………………………
9) $3(x - 2)(x + 1)$ = …………………………………………………………………………………
………………………………………………………………………………………………………
10) $6x(x - 1)^2$ = ……………………………………………………………………………………
………………………………………………………………………………………………………
11) $(3x + 2)^2 - (x^2 - 2)$ = ……………………………………………………………………………
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§ 4.6 THE PRODUCT OF A BINOMIAL AND A TRINOMIAL

- A **trinomial** has THREE terms. "tri" means three.

  The following are examples of trinomials:
  $x^2 + 2x - 1$;
  $ax^2 + bx + c$
  $\frac{3}{4}a + 27b - 0{,}25c$

A **tri**cycle has three wheels

- To simplify an example like $(x + 3)(x^2 + 2x - 1)$,
  - multiply the first term in the first bracket with each term of the second bracket
  - multiply the second term in the first bracket with each term of the second bracket

$$\mathbf{(x + 3)(x^2 + 2x - 1) = x^3 + 2x^2 - x + 3x^2 + 6x - 3 = x^3 + 5x^2 + 5x - 3}$$

EXAMPLE	SOLUTION
Simplify $(2x - 1)(x^2 + 2x - 1)$	$(2x - 1)(x^2 + 2x - 1)$ $= 2x(x^2 + 2x - 1) - 1(x^2 + 2x - 1)$ $= 2x^3 + 4x^2 - 2x - x^2 - 2x + 1$ $= 2x^3 + 3x^2 - 4x + 1$

Exercise 4.6

Remove the brackets and simplify:

1) $(x + 1)(x^2 + 3x + 5) =$ ……………………………………………………………………

……………………………………………………………………………………………

2) $(1 - x)(x^2 - x - 2) =$ ……………………………………………………………………

……………………………………………………………………………………………

3) $(x - 2)(x^2 - 4x - 3) =$ ……………………………………………………………………

……………………………………………………………………………………………

4) $(2x^2 + x - 1)(x - 1) =$ ……………………………………………………………………

……………………………………………………………………………………………

5) $(3x - 4)(x^2 + 2x - 1) =$ ……………………………………………………………………

……………………………………………………………………………………………

6) $(2x + 1)(3x^2 + x - 2) =$ ……………………………………………………………………

……………………………………………………………………………………………

7) $(3x - 2)(2x^2 - 3x - 5) =$ ……………………………………………………………………

……………………………………………………………………………………………

8) $(3a - 2b)(5a^2 + 4ab - 3b^2) =$ ……………………………………………………………

……………………………………………………………………………………………

§ 4.7 COMMON FACTORS

- When we find a **product** we multiply.
  Example 1: The product of 2 and 3 is $2 \times 3 = 6$
  Example 2: The product of 3 and $(2x + 1)$ is $3(2x + 1) = 6x + 3$.
- When we **factorise** we break a product up into its **factors**.
  Example 3: When we factorise 6 we get 2×3, which we can also write (2)(3)
  Example 4: When we factorise $6x + 3$, we first find the factor that is common to $6x$ and to 3. This is 3. So $6x + 3 = 3(2x + 3)$

Exercise 4.7

1) Factorise:

a) $3x + 6y =$ ……………………………..……..

b) $4p - 2q =$ ……………………………………

c) $x^2 + xy =$ ……………………..…………...

d) $3x - 5xy^3 =$ …………………………………

e) $5a + 10b - 15c =$ ……………………….......

f) $-3x + 15y =$ ………………………………...

- Sometimes there is more than one factor which is common to all the terms. Always take out the **highest common factor**, the HCF.

EXAMPLE		SOLUTION
1)	Factorise $12x + 6y + 18$	2 is a common factor, so $12x + 6y + 18 = 2(6x + 3y + 9)$. But 3 is also a common factor, so $12x + 6y + 18 = 3(4x + 2y + 6)$ And 6 is also a common factor, so $12x + 6y + 18 = 6(2x + y + 3)$ Always take out the *Highest Common Factor (HCF).* So $12x + 6y + 18 = 6(2x + y + 3)$
2)	Factorise: $a^3b^2c^4 - 2a^2b^3c^2 + a^3bc^3$	The HCF of $a^3b^2c^4$; $2a^2b^3c^2$ and a^3bc^3 is a^2bc^2 So $a^3b^2c^4 - 2a^2b^3c^2 + a^3bc^3$ $= a^2bc^2(abc^2 - 2ab^2 + ac)$

2) Factorise by taking out the Highest Common Factor (HCF):

a) $18x + 12y =$ ………………………………..

b) $24p - 12q =$ ……………………………..…

c) $6a^3 + 3a^2 =$ …………………………..….

d) $4xy - 2y =$ ………………………………….

e) $27m^3 - 9m^2 =$ ……………………………….

f) $9xy^2 - 3x^2y =$ ………………………………..

§ 4.8 BINOMIALS AS COMMON FACTORS

✧ Binomials and trinomials can also be common factors.

Example 1: $2x\underline{(x-2)} + 3\underline{(x-2)} = \underline{(x-2)}(2x+3)$. The underlined bracket is the common factor.

Example 2: $\underline{(x+y)} - 2a\underline{(x+y)} - b\underline{(x+y)} = \underline{(x+y)}(1-2a-b)$

Exercise 4.8

1) Factorise the following:

a) $2(x-4) + 3x(x-4) =$ ……………………………………………………………………

b) $9y(2x+3) + 5(2x+3) =$ ……………………………………………………………………

c) $(2x-1) - 3p(2x-1) =$ ……………………………………………………………………

d) $a + b - 2b(a+b) =$ ……………………………………………………………………

e) $2x(x^2 - x + 1) + y(x^2 - x + 1) =$ ……………………………………………………………………

f) $3a(a+b-c) - (a+b-c)4b =$ ……………………………………………………………………

g) $2b(c+2d) - 3d(c+2d) =$ ……………………………………………………………………

h) $6c(a+b) + p(b+a) =$ ……………………………………………………………………

✧ We can change the order of the terms of a binomial.

Example 3: $(b+a) = (a+b)$

Example 4: $(b-a) = -a + b = -(a-b)$. We have taken out a common factor of -1

Example 5: $d - 2c = -2c + d = -(2c - d)$

Example 6: $3d(p-q) + 2q(q-p)$

$= 3d(p-q) + 2q(-p+q) = 3d\underline{(p-q)} - 2q\underline{(p-q)} = \underline{(p-q)}(3d - 2q)$

2) Factorise:

a) $2(b-c) + b(c-b) =$ ……………………………………………………………………

b) $3(2-5a) + x(5a-2) =$ ……………………………………………………………………

c) $3n(2a-b) - 4(b-2a) =$ ……………………………………………………………………

d) $x(2x-y) + y(y-2x) =$ ……………………………………………………………………

e) $y^2(3b-a) - (a-3b) =$ ……………………………………………………………………

f) $5b(-a+b) + 6c(b-a) =$ ……………………………………………………………………

g) $2x(3x+y) - 5y(y+3x) =$ ……………………………………………………………………

h) $4a(3b-2c) + 2(2c-3b) - y(3b-2c) =$ ……………………………………………………………………

§ 4.9 GROUPING TO FIND A COMMON FACTOR

✧ Expressions with three or more terms can often be factorised by "grouping terms" in order to find a common factor.

Example 1: $ac + ad + bc + bd$
- Group the first 2 terms and take out a common factor of a

$= a(c + d) + b(c + d)$
- Group the second 2 terms and take out a common factor of b

$= (c + d)(a + b)$
- Factorise as you did on the previous page.

Example 2: $xy - 6ab + 2ay - 3bx$
- Study the four terms and decide which ones go together.

$= xy + 2ay - 3bx - 6ab$
- Group the 1st and 3rd terms, then group the 4th and 2nd terms.

$= y(x + 2a) - 3b(x + 2a)$
- Take out a common factor of y and then a common factor of $-3b$.

$= (x + 2a)(y - 3b)$
- Factorise as you did on the previous page.

Exercise 4.9

Factorise the following:

1) $2ax + 2ay + bx + by$ = ……………………………………………………………………

……………………………………………………………………………………………

2) $6ab - 4bd + 9ac - 6cd$ = ……………………………………………………………………

……………………………………………………………………………………………

3) $2ax - ay + 6bx - 3by$ = ……………………………………………………………………

……………………………………………………………………………………………

4) $wx - yw + xz - yz$ = ……………………………………………………………………

……………………………………………………………………………………………

5) $6wx - 10xz + 15yw - 25yz$ = ……………………………………………………………………

……………………………………………………………………………………………

6) $2 + 2u - a - ua + b + ub$ = ……………………………………………………………………

……………………………………………………………………………………………

7) $12a^2 - 25byz - 10ayz + 30ab$ = ……………………………………………………………………

……………………………………………………………………………………………

8) $2ac + 3ad + a - 2bc - 3bd - b$ = ……………………………………………………………………

……………………………………………………………………………………………

§ 4.10 DIFFERENCE OF TWO SQUARES

Exercise 4.10

1) Simplify the following:

   a) $(x - 1)(x + 1) =$ ……………………………………………………………………………….......

   b) $(x + 3)(x - 3) =$ ………………………………………………………………………………

   c) $(2x - 5)(2x + 5) =$ ……………………………………………………………………………….

2) Study the answers. What is the same about them?

   ……………………………………………………………………………………………………….

- The word "difference" means the answer to a subtraction of two terms.
  $4 = 2 \times 2 = 2^2$, $9 = 3 \times 3 = 3^2$, $16 = 4 \times 4 = 4^2$ etc. So 4, 9 and 16 are perfect squares.
- In the difference of two squares, **each term** must be a **perfect square**.
  $4a^2 - 9 = (2a)^2 - 3^2$ is a difference of two squares, which factorises as $(2a + 3)(2a - 3)$.
  Check this answer by multiplying out: $(2a + 3)(2a - 3) = 4a^2 - 6ab + 6ab - 9 = 4a^2 - 9$
- In order for the signs of the Inners and Outers to be different and cancel out, the sign in the each bracket must be different.
- Factorise $p^2 - q^2$. This is a difference of two squares, so we simply use **F**irsts and **L**asts in the brackets, as the **O**uters and **I**nners cancel out. So

$$\boldsymbol{p^2 - q^2 = (p - q)(p + q) \text{ or } (p + q)\,(p - q)}$$

Note:
$x^2 + 1$ does not factorise as it is not a **difference** of two squares. It is a **sum** of two squares.

EXAMPLE		**SOLUTION**
	Factorise	
1)	$4a^2 - b^2$	$4a^2 - b^2 = (2a)^2 - b^2 = (2a - b)(2a + b)$
2)	$p^6 - 36$	$p^6 - 36 = (p^3 - 6)(p^3 + 6)$

3) Factorise the following:

   a) $x^2 - 4 =$ ………………………………………………………………………………

   b) $a^2 - b^2 =$ ……………………………………………………………………………..

   c) $4x^2 - 9 =$ ……………………………………………………………………………….....

   d) $y^2 - 1 =$ ………………………………………………………………………………

   e) $25a^2 - 16b^2 =$ ……………………………………………………………………………..

   f) $a^4b^2 - 1 =$ ………………………………………………………………………………

   g) $16p^2 + 4 =$ ………………………………………………………………………………

   h) $1 - 4b^2 =$ ………………………………………………………………………………

   i) $p^2 - 81q^2 =$ ……………………………………………………………………………..

   j) $x^6 - 16 =$ ……………………………………………………………………………..

§ 4.11 FACTORISING TRINOMIALS WITH TWO POSITIVE SIGNS

- When factorising trinomials, we first look at the two signs (+ or –) of the trinomial.
- Remember that the sign belongs to the number that comes after it.
- STEP 1:
  Look at the second sign.
  If the **second sign** is positive (+), the signs in the two brackets will be the same.
- STEP 2:
  Look at the first sign
  If the **first sign** is positive (+), both signs will be positive (+)
  If the **first sign** is negative (–), both signs will be negative (–)
- STEP 3:
  Find the product of the 3rd terms of the trinomial, that **added** will give the 2nd term.
  Write these two factors in the brackets.

Example: Factorise $x^2 + 5x + 6$

- The last sign is positive, which tells us that both signs are the same.
  The first sign is positive, which tells us that both signs are positive
  This means we have $(x + \ldots..)(x + \ldots..)$
- We need the factors of 6 that **added** give us 5.
  Possible factors are 1×6 or 2×3. But $1 + 6 = 7$, and $2 + 3 = 5$
  This tells us we have to use 2 and 3. So $x^2 + 5x + 6 = \mathbf{(x + 2)(x + 3)}$

EXAMPLE		SOLUTION
Factorise:		
1)	$x^2 + 2x + 1$	$x^2 + 2x + 1 = (x + 1)(x + 1)$ *The only factors of 1 are 1×1. $1 + 1 = 2$, so 1 and 1 are the correct factors.*
2)	$x^2 + 5x + 4$	$x^2 + 5x + 4 = (x + 1)(x + 4)$ *The factors of 4 are 2×2 and 1×4* *$2 + 2 = 4$ and $1 + 4 = 5$, so 1 and 4 are the correct factors.*

Exercise 4.11

Factorise:

1) $x^2 + 3x + 2 =$ …………………………………………………………………………………..

2) $x^2 + 8x + 7 =$ …………………………………………………………………………………..

3) $x^2 + 9x + 8 =$ ………………..……………………………………………………………………

4) $x^2 + 6x + 8 =$ ………………………….………………………………………………………..

5) $x^2 + 6x + 9 =$ …………………………………………………………………………………..

6) $x^2 + 10x + 9 =$ ……………………………………………………………………………….

7) $x^2 + 7x + 10 =$ ………………………………………………………………………………

8) $x^2 + 11x + 10 =$ …………………………………………………………………………..

9) $x^2 + 13x + 12 =$ …………………………………………………………………………..

10) $x^2 + 7x + 12 =$ ………………………………………………………………………………

§ 4.12 FACTORISING TRINOMIALS WITH A NEGATIVE AND A POSITIVE SIGN

- When we factorise and the **last sign** is positive (+), the signs in the two brackets will be **the same**.
- If the **first sign** is positive (+), they will both be positive (+)
  If the **first sign** is negative (–), they will both be negative (–)

Example: Factorise $x^2 - 5x + 4$

- The **last sign** is positive, which tells us that both signs are the same.
  The **first sign** is negative, which tells us that both signs are negative
  This means we have $(x - \ldots\ldots)(x - \ldots\ldots)$
- We need the factors of 4 that **added** give us 5.
  Possible factors are 1×4 or 2×2
  $1 + 4 = 5$, and $2 + 2 = 4$ – this tells us we have to use 1 and 4.
  So $x^2 - 5x + 4 = \mathbf{(x - 2)(x - 2)}$

EXAMPLE		SOLUTION
Factorise:		
1)	$x^2 - 4x + 4$	$x^2 - 4x + 4 = (x - 2)(x - 2)$ *The factors of 4 are 1×4 or 2×2.* *$1 + 4 = 5$ and $2 + 2 = 4$. So 2 and 2 are the correct factors.*
2)	$x^2 - 5x + 6$	$x^2 - 5x + 6 = (x - 2)(x - 3)$ *The factors of 6 are 1×6 or 2×3* *$1 + 6 = 7$ and $2 + 3 = 5$. So 2 and 3 are the correct factors.*

<u>Exercise 4.12</u>

Factorise:

1) $x^2 - 4x + 3 =$ ……………………………………………………………………………………..
2) $x^2 - 3x + 2 =$ ……………………………………………………………………………………..
3) $x^2 - 5x + 4 =$ ……………………………………………………………………………………..
4) $x^2 - 6x + 5 =$ ……………………………………………………………………………………..
5) $x^2 - 7x + 6 =$ ……………………………………………………………………………………..
6) $x^2 + 12x + 11 =$ …………………………………………………………………………………..
7) $x^2 - 6x + 8 =$ ……………………………………………………………………………………..
8) $x^2 - 9x + 8 =$ ……………………………………………………………………………………..
9) $x^2 + 6x + 9 =$ ……………………………………………………………………………………..
10) $x^2 + 10x + 9 =$ ……………………………………………………………………………………
11) $x^2 - 7x + 12 =$ ……………………………………………………………………………………
12) $x^2 - 8x + 12 =$ ……………………………………………………………………………………
13) $x^2 + 8x + 12 =$ ……………………………………………………………………………………
14) $x^2 + 14x + 13 =$ …………………………………………………………………………………

§ 4.13 FACTORISING TRINOMIALS WHEN THE LAST SIGN IS NEGATIVE

- When we factorise and the **last sign** is negative (–), the signs in the two brackets will be **different** and we have to **subtract** the two factors

Example: Factorise $x^2 + 4x - 5$

- We need the factors of 5 that **subtracted** give us 4.
  The only possible factors are 1×5
  $5 - 1 = 4$. This tells us that the required factors are 1 and 5.
- The **last sign** is negative, which tells us that both signs are the different
  The **first sign** is positive, which tells us that the larger of the **inner** and **outer** is positive.
  So $x^2 + 4x - 5 = \boldsymbol{(x + 5)(x - 1)}$

EXAMPLE		SOLUTION
Factorise:		
1)	$x^2 - 3x - 4$	$x^2 - 3x - 4 = (x - 4)(x + 1)$ • *The last sign is negative, so the factors of 4 are* ***subtracted*** *to give 3.* • *The factors of 4 = 2 × 2 or 1 × 4* *2 – 2 = 0 and 4 – 1 = 3. So 1 and 4 are the correct factors* • *The first sign is negative, meaning the larger product is* ***negative.***
2)	$x^2 + 3x - 4$	$x^2 + 3x - 4 = (x + 4)(x - 1)$ • *The last sign is negative, so the factors of 4 are* ***subtracted*** *to give 3.* • *The factors of 4 = 2 × 2 or 1 × 4* *2 – 2 = 0 and 4 – 1 = 3. So 1 and 4 are the correct factors* • *The first sign is positive, meaning the larger product is* ***positive.***

<u>Exercise 4.13</u>

Factorise:

1) $x^2 + 2x - 3 =$ ……………………………………………………………………………………

2) $x^2 - 2x - 3 =$ ……………………………………………………………………………………

3) $x^2 + 6x - 7 =$ ……………………………………………………………………………………

4) $x^2 - 6x - 7 =$ ……………………………………………………………………………………

5) $x^2 - 4x - 5 =$ ……………………………………………………………………………………

6) $x^2 + 6x + 5 =$ ……………………………………………………………………………………

7) $x^2 + 2x - 8 =$ ……………………………………………………………………………………

8) $x^2 - 2x - 8 =$ ……………………………………………………………………………………

9) $x^2 - 7x - 8 =$ ……………………………………………………………………………………

10) $x^2 + 9x + 8 =$ ……………………………………………………………………………………

11) $x^2 - 8x - 9 =$ ……………………………………………………………………………………

12) $x^2 + 8x - 9 =$ ……………………………………………………………………………………

13) $6x^2 + 7x + 1 =$ ……………………………………………………………………………………

14) $6x^2 - x - 1 =$ ……………………………………………………………………………………

§ 4.14 FACTORISING MORE TRINOMIALS

- If we simplify (2x + 3)(3x + 5), we obtain $6x^2 + 19x + 15$.
  NOTICE that
  - The 1st term of the answer is made up from $2x \times 3x$
  - The 3rd terms of the answer is made up from 3×5
  - The middle term is made up from $(2x \times 5) + (3x \times 3)$

- So if we have to factorise $6x^2 + 19x + 15$, we have to find the factors of $6x^2$ and 15 which give us the middle term + 19x.
  - The factors of 6 are 1×6 and 2×3
  - The factors of 15 are 1×15 and 3×5

  We have to take these factors and try out the combinations until we find the right one.

Factors of 6	Factors of 15
1	1
15	6

15 + 6 = 21 – *NO*

Factors of 6	Factors of 15
1	1
15	6

1 + 90 = 91 – *NO*

Factors of 6	Factors of 15
2	1
15	3

30 + 3 = 33 – *NO*

Factors of 6	Factors of 15
2	1
15	3

2 + 45 = 47 – *NO*

Factors of 6	Factors of 15
2	3
5	3

10 + 9 = 19 – *YES*

Factors of 6	Factors of 15
2	3
5	3

6 + 15 = 21 – *NO*

- Write down the factors of $6x^2$ first: (2x ……..)(3x ……..)

- Use your working above to write the factors of 15 in the correct place: (2x …3)(3x …5)

- Add the signs: (2x + 3)(3x + 5)

- Check using FOIL: (2x + 3)(3x + 5) = $6x^2 + 10x + 9x + 15 = 6x^2 + 19x + 15$

EXAMPLE		SOLUTION
Factorise:		
1)	$2x^2 + 5x + 2$	$2x^2 + 5x + 2 = (2x + 1)(x + 2)$ • *The last sign is **positive**, so the factors of 2 and 2 are **added** to give 5.* • *The factors of 2 = 1 × 2.* *Try all combinations of factors until the correct one is found.* • *The first sign is positive, meaning both products are **positive.***
2)	$2x^2 - 7x + 6$	$2x^2 - 7x + 6 = (2x - 3)(x - 2)$ • *The last sign is **positive**, so the factors of 2 and 6 are **added** to give 7.* • *The factors of 2 = 1 × 2 and the factors of 6 = 1 × 6 or 2 × 3* *Try all combinations of factors until the correct one is found.* • *The first sign is negative, meaning both products are **negative.***
3)	$3x^2 + 10x - 8$	$3x^2 + 10x - 8 = (3x - 2)(x + 4)$ • *The last sign is **negative**, so the factors of 3 and 8 are **subtracted** to give 10.* • *The factors of 3 = 1 × 3 and the factors of 8 = 1 × 8 or 2 × 4* *Try all combinations of factors until the correct one is found.* • *The first sign is positive, meaning the larger product is **positive.***

Exercise 4.14

Factorise fully:

1) $2x^2 - 5x + 2 =$ ……………………………………………………………………

……………………………………………………………………

2) $3x^2 - 11x - 4 =$ ……………………………………………………………………

……………………………………………………………………

3) $2x^2 + 3x - 2 =$ ……………………………………………………………………

……………………………………………………………………

4) $5x^2 - 9x - 2 =$ ……………………………………………………………………

……………………………………………………………………

5) $3x^2 - 13x + 4 =$ ……………………………………………………………………

……………………………………………………………………

6) $5x^2 - 21x - 20 =$ ……………………………………………………………………

……………………………………………………………………

7) $3x^2 + x - 4 =$ ……………………………………………………………………

……………………………………………………………………

8) $7x^2 - 30x + 8 =$ ……………………………………………………………………

……………………………………………………………………

9) $11x^2 - 12x + 1 =$ ……………………………………………………………………

……………………………………………………………………

10) $7x^2 - 15x + 2 =$ ……………………………………………………………………

……………………………………………………………………

§ 4.15 DIVISION BY A MONOMIAL

- The answer to a division calculation is called a **quotient**.

  Example 1: The quotient of 6 and 2 is $\frac{6}{2}$ which is 3

  Example 2: The quotient of x and y is $\frac{x}{y}$.
- When we divide, we cancel factors which are common to the numerator and the denominator. If we have more than one term in the numerator, we must factorize before we cancel.

EXAMPLE		SOLUTION
Divide and simplify:		
1)	$\frac{2x^2y^3}{4xy}$	$\frac{2x^2y^3}{4xy} = \frac{xy^2}{2}$
2)	$\frac{2x+6y}{2}$	$\frac{2x+6y}{2} = \frac{2(x+3y)}{2} = x + 3y$
3)	$\frac{2a^2-4a}{2a}$	$\frac{2a^2-4a}{2a} = \frac{2a(a-2)}{2a} = a - 2$

<u>Exercise 4.15</u>

Divide and simplify:

1) $\frac{8x^3}{4x^2}$ = ……………………………………………………………………………………………

2) $\frac{9a^3bc^2}{3a^2bc^2}$ = ……………………………………………………………………………………………

3) $\frac{4p^3}{12p^2}$ = ……………………………………………………………………………………………..

4) $\frac{2c^3d^2}{6cd^4}$ = ……………………………………………………………………………………………

5) $\frac{4^2.3^3}{4^3.3^2.9}$ = ……………………………………………………………………………………………

6) $\frac{ab+a}{a}$ = ……………………………………………………………………………………………

7) $\frac{4a^2+2a}{2a}$ = ……………………………………………………………………………………………..

8) $\frac{6s+3s^2}{3s}$ = ……………………………………………………………………………………………

9) $\frac{6x+12y-3z}{3}$ = ……………………………………………………………………………………………

10) $\frac{8b^3+2b}{2b}$ = ……………………………………………………………………………………………..

11) $\frac{8t^3-4t^2}{4t^2}$ = ……………………………………………………………………………………………..

CHAPTER 5
LINEAR EQUATIONS AND LINEAR FUNCTIONS

In this Unit you will

- Solve linear equations by adding a number or expression to both sides
- Solve linear equations by multiplying or dividing by the same number
- Solve equations containing fractions
- List numbers that satisfy a given inequality
- Solve inequalities by adding, subtracting, multiplying or dividing by a number
- Plot points on a grid
- Calculate x and y values using a given equation
- Find values using function notation
- Draw the graph of a straight line from an equation
- Draw the graph of a straight line by finding the x- and y-intercept
- Investigate the characteristics of straight lines
- Calculate the slope of a line
- Work out the gradient of a straight line
- Use a formula to work out the gradient
- Compare the gradients of parallel and perpendicular lines
- Draw straight lines using the gradient-intercept method
- Find the equation of a straight line given the y-intercept and the co-ordinates of one other point
- Find the equation of a straight line given the co-ordinates of two points on the line
- Find the equation of vertical and horizontal lines
- Use graphs to find the point of intersection of two straight lines
- Find the intersection of two lines algebraically
- Use function notation to find the gradient of a straight line
- Find the average gradient between two points on a curve

This chapter covers material from Topic 2: Functions

SUBJECT OUTCOME 2.3:
Solve algebraic equations

Learning Outcome 1:	Solve linear equations.
Learning Outcome 4:	Solve inequalities in one variable graphically.
Learning Outcome 5:	Solve equations simultaneously by numerical, algebraic and graphical methods.

SUBJECT OUTCOME 2.1:
Work with a range of functions and patterns and solve problems

Learning Outcome 1:	Sketch and interpret $y = ax + q$.
Learning Outcome 2:	Convert between numerical, verbal, symbolic and graphical representations of information.
Learning Outcome 3:	Investigate and generalise the impact of a and q in the functions above.

SUBJECT OUTCOME 2.4:
Investigate the average rate of change of a function between two values of the independent variable

Learning Outcome 1:	Investigate the gradient between two points on a curve with specific reference to linear functions.
Learning Outcome 2:	Investigate the gradient of a curve at a specific point.

§ 5.1 SOLVING LINEAR EQUATIONS BY ADDING A NUMBER OR EXPRESSION TO BOTH SIDES

- An equation is a mathematical statement involving an "=" sign.
  Example: $2x - 3y = 5$; $2x^2 - x + 5 = 0$; $xy = 0$.
- Any value of the variable which makes the equation true is called a **solution** of the equation.
- The solution of an equation remains unchanged when the same number or expression is added to both sides.

EXAMPLES	
1) Solve $7 + x = 16$ $7 + x = 16$ $7 + x - 7 = 16 - 7$ $x = 9$ **Check:** Substitute $x = 9$ into the left hand side and right hand side of the equation LHS = $7 + 9 = 16$ RHS = 16 LHS = RHS	2) Solve $15 = a - 2$ $15 = a - 2$ $15 + 2 = a - 2 + 2$ $17 = a$ **Check:** LHS = 15 RHS = $17 - 2 = 15$ LHS = RHS

<u>Exercise 5.1</u>

Solve each of the following equations and check your answers:

1) $d - 12 = 16$
2) $p + 12 = 18$
3) $3x - 5 = 7 + 2x$
4) $5x + 4 = 9 + 4x$
5) $18 - 3x = -4x - 10$
6) $3x - 2 + x = 3x - 5$
7) $8x + 9 - 3x = 4x + 2$
8) $5 - x = 3 - 2x$
9) $2(4 - x) = 3(6 - x)$

§ 5.2 SOLVING LINEAR EQUATIONS BY MULTIPLYING OR DIVIDING BY THE SAME NUMBER

- The solution of an equation remains unchanged if both sides of the equation are divided or multiplied by the same number

EXAMPLES	
1) Solve $5p = 15$ $5p = 15$ $\frac{5p}{5} = \frac{15}{5}$ $p = 3$ **Check:** LHS = $5 \times 3 = 15$ RHS = 15 LHS = RHS	2) Solve $\frac{a}{4} = 5$ $\frac{a}{4} = 5$ $\frac{a \times 4}{4} = 4 \times 5$ $a = 20$ **Check:** LHS = $\frac{20}{4} = 5$ RHS = 5 LHS = RHS

Exercise 5.2

Solve each of the following equations and check your answers:

1) $3x = 21$
2) $5x = 35$
3) $3x = 1$
4) $6x = 9$
5) $\frac{x}{2} = 3\frac{1}{2}$
6) $\frac{t}{6} = 5$
7) $\frac{x}{5} = 3$
8) $\frac{3x}{4} = 15$
9) $2x = 0$

§ 5.3 SOLVING LINEAR EQUATIONS BY ADDING AND BY DIVIDING OR MULTIPLYING

EXAMPLES		
1) Solve $2x - 1 = 5$ $2x - 1 = 5$ $2x - 1 + 1 = 5 + 1$ $2x = 6$ $\frac{2x}{2} = \frac{6}{2}$ $x = 3$	2) Solve $2 - 3p = 17$ $2 - 3p = 17$ $2 - 3p - 2 = 17 - 2$ $-3p = 15$ $\frac{-3p}{-3} = \frac{15}{-3}$ $p = -5$	3) Solve $\frac{a}{3} + 4 = 9$ $\frac{a}{3} + 4 = 9$ $\frac{a}{3} + 4 - 4 = 9 - 4$ $\frac{a}{3} = 5$ $\frac{a}{3} \times 3 = 5 \times 3$ $a = 15$

Exercise 5.3

Solve:

1) $6d + 1 = 19$

2) $6p - 7 = 23$

3) $\frac{x}{3} + 1 = 8$

4) $16 - 3x = 4$

5) $3(y - 3) = 18$

6) $13 = 34 - y$

7) $3a - 7a + 5 = 0$

8) $12 - 4b = b - 13$

9) $2(n - 4) + 3(n + 2) = 0$

§ 5.4 SOLVING LINEAR EQUATIONS CONTAINING FRACTIONS

EXAMPLES		
1) Solve $\frac{a}{4}+\frac{a}{5}=\frac{9}{2}$ $\frac{a}{4}+\frac{a}{5}=\frac{9}{2}$ *Multiply each term by the LCM of the denominators* $\left(\frac{a}{4}\times 20\right)+\left(\frac{a}{5}\times 20\right)=\frac{9}{2}\times 20$ $5a + 4a = 90$ $9a = 90$ $\frac{9a}{9}=\frac{90}{9}$ $a = 10$	2) Solve $\frac{(x+1)}{2} = 9$ $\frac{(x+1)}{2} = 9$ $\frac{(x+1)}{2} \times 2 = 9 \times 2$ $x + 1 = 18$ $x + 1 - 1 = 18 - 1$ $x = 17$	3) Solve $\frac{5k+3}{4} = 11$ $\frac{5k+3}{4} = 11$ $\frac{(5k+3)}{4} \times 4 = 11 \times 4$ $5k + 3 = 44$ $5k + 3 - 3 = 44 - 3$ $5k = 41$ $\frac{5k}{5} = \frac{41}{5}$ $k = 8\frac{1}{5}$

Exercise 5.4

Solve each of the following equations:

1) $\frac{x}{3}+\frac{x}{2}=\frac{5}{2}$

2) $\frac{x}{6}+\frac{2x}{3}=\frac{5}{4}$

3) $\frac{x}{5}-\frac{2x}{3}=3\frac{1}{2}$

4) $\frac{4x}{5}-\frac{9}{4}=\frac{x}{2}$

5) $\frac{2x}{7}+\frac{x}{2}=\frac{4}{2}$

6) $\frac{2b}{3} + 3 = 9$

7) $\frac{2}{3}(d+1) = 8$

8) $\frac{6-4q}{7} = 4$

9) $\frac{3(2x+3)}{5} = 45$

§ 5.5 INEQUALITIES

- An inequality is a statement involving "less than" (<) or "greater than" (>) or "less than or equal to" (≤) or "greater than or equal to" (≥) or "not equal to" (≠).
- Number sentences expressing these relationships are called inequalities.
- Inequalities between numbers can be shown on a number line. The numbers increase going from left to right.

–5 –4 –3 –2 –1 0 1 2 3 4 5

EXAMPLES		
–3 < –1	–2 < 3	1 < 4
4 > 2	1 > -1	–1 > -4

Exercise 5.5

From the list of numbers –10; –9; –8; –7; –6; –5; –4; –3; –2; –1; 0; 1; 2; 3; 4; 5; 6; 7; 8; 9; 10, write down

12) The numbers greater than 6……………………………………………………………………

……………………………………………………………………………………………………

13) The numbers greater than 0……………………………………………………………………

……………………………………………………………………………………………………

14) The numbers less than 4………………………………………………………………………

……………………………………………………………………………………………………

15) The numbers less than – 4 ……………………………………………………………………

……………………………………………………………………………………………………

16) The numbers less than –7 ……………………………………………………………………

……………………………………………………………………………………………………

17) The numbers greater than –1 …………………………………………………………………

……………………………………………………………………………………………………

18) The numbers which are less than – 10 ………………………………………………………

……………………………………………………………………………………………………

19) The numbers which are not equal to 0…………………………………………………………

……………………………………………………………………………………………………

20) The numbers which are not less than 0…………………………………………………………

……………………………………………………………………………………………………

21) The numbers which are not greater than 0………………………………………………………

……………………………………………………………………………………………………

§ 5.6 SOLVING INEQUALITIES BY ADDING OR SUBTRACTING FROM BOTH SIDES

✧ The solution of an inequality remains unaltered if the same number or expression is added to or subtracted from each side.

EXAMPLES	
1) We know that $8 > 6$ Suppose we add 5 to both sides We get $8 + 5 > 6 + 5$ $13 > 11$ which is true Now suppose we subtract 7 from both sides We get $13 - 7 > 11 - 7$ $6 > 4$ which is true	2) Solve $3a - 2 > 2a + 4$ for a $3a - 2 > 2a + 4$ $3a - 2 - 2a > 2a + 4 - 2a$ *(Add –2a to both sides)* $a - 2 > 4$ $a - 2 + 2 > 4 + 2$ *(Add 2 to both sides)* $a > 6$

Exercise 5.6

Solve the following inequalities

1) $2m + 3 < m - 5$

2) $5p > 6 + 4p$

3) $7 - 3k < 1 - 4k$

4) $13 - 7q < q + 7 - 9q$

5) $11u > 2(2 + 5u)$

6) $2u - 3 < u - 3$

§ 5.7 SOLVING INEQUALITIES BY MULTIPLYING BY OR DIVIDING BY A NUMBER

- The solution of an inequality remains unaltered if both sides are multiplied by or divided by a **positive whole number**.
- The relationship changes direction when both sides of the inequality are multiplied or divided by a **negative number**. So "is greater than" changes to "is less than" (and vice versa)

EXAMPLES	
1) We know that $4 < 10$ • Suppose we divide each side by 2 We get $2 < 5$ which is true • Suppose we multiply each side by –3 We get $-6 < -15$ **which is NOT true** In fact, $-6 > -15$ • Suppose we multiply each side by –2 We get $+12 > +30$ **which is NOT true**	2) Solve $b - 5 > 3b + 9$ $b - 5 > 3b + 9$ *(add –3b to both sides)* $b - 5 - 3b > 3b + 9 - 3b$ $-2b - 5 > 9$ $-2b - 5 + 5 > 9 + 5$ *(add 5 to both sides)* $-2b > 14$ *(divide by –2 and change the direction of the inequality)* $\frac{-2b}{-2} < \frac{14}{-2}$ $b < -7$

Exercise 5.7

Solve the inequalities:

1) $4a > 6a - 2$
2) $19 - 3a > 2a - 1$
3) $9 - 2a < 1$

4) $14 - a > 0$
5) $11 - 4a < -1$
6) $2a + 9 > 5a - 3$

§ 5.8 SOLVING INEQUALITIES BY ADDING AND BY MULTIPLYING

EXAMPLES		
1) Solve $2x - 3 \geq 6(x + 1)$ $2x - 3 \geq 6(x + 1)$ *(multiply out)* $2x - 3 \geq 6x + 6$ *(add –2x to both sides)* $2x - 3 - 2x \geq 6x - 2x + 6$ $-3 \geq 4x + 6$ *(add –6 to both sides)* $-3 - 6 \geq 4x + 6 - 6$ $-9 \geq 4x$ *(divide each side by 4)* $\frac{-9}{4} \geq \frac{4x}{4}$ $\frac{-9}{4} \geq x$	2) Solve $\frac{-x}{3} < -5$ $\frac{-x}{3} < -5$ *(multiply each side by 3)* $\frac{-x}{3} \times 3 < -5 \times 3$ $-x < -15$ *(the sign changes when we divide each side by -1)* $\frac{-x}{-1} > \frac{-15}{-1}$ $x > 15$	3) Solve $-3 \leq 1 - 2x < 7$ $-3 \leq 1 - 2x < 7$ *(add –1 to both sides)* $-3 - 1 \leq 1 - 2x - 1 < 7 - 1$ $-4 \leq -2x < 6$ *(the signs change when we divide by –2)* $\frac{-4}{-2} \geq \frac{-2x}{-2} > \frac{6}{-2}$ $2 \geq x > -3$

Exercise 5.8

Solve the inequalities:

1) $2x + 1 > 7$

2) $3 - 4x \leq x - 2$

3) $-2 \leq 4 - 2x < 12$

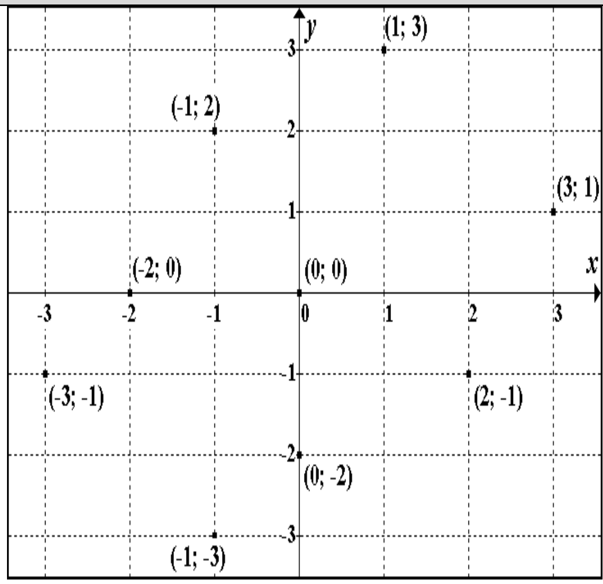
4) $1 - 2x > x - 2$

5) $\frac{x}{2} \geq -3x + 7$

6) $\frac{(y + 3)}{2} + 5 > 8$

§ 5.9 PLOTTING POINTS ON A GRID

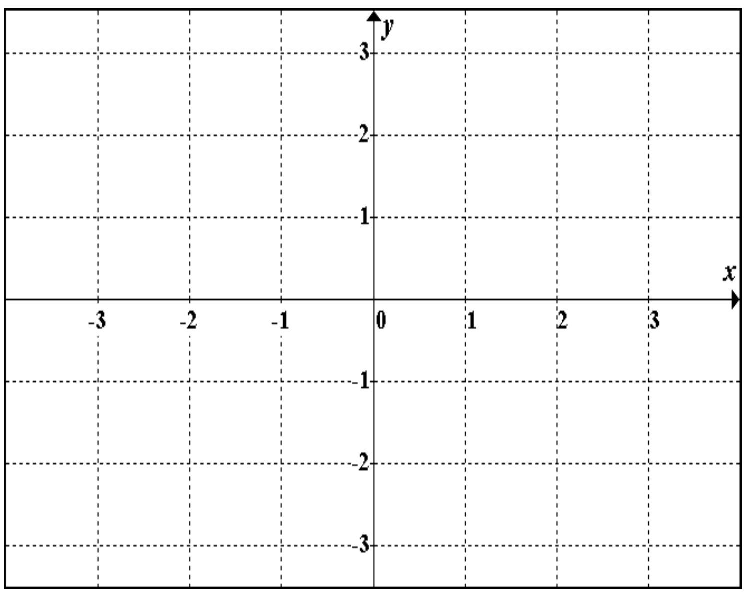
✧ The **vertical** axis is the ***y*-axis** and the **horizontal** axis is the ***x*-axis**. Every point has coordinates (x; y). The x-coordinate is always written first. The x-coordinate value is measured along the x-axis and the y-coordinate value is measured along the y-axis.

EXAMPLE	SOLUTION
On a grid, plot each of the following: (0; 0) (1; 3) (3; 1) (–1; 2) (2; –1) (–3, –1) (–1, –3)	

Note:
- Points (1; 3) and (3; 1) have different positions on the plane.
- The point (0; 0) lies where the x and y axes intersect and is called the **origin.**

Exercise 5.9

1) Write down the coordinates of the origin: ………..……..........
2) Which axis is vertical?
   …………………...……..........
3) Which axis is horizontal?
   ……………………......……...
4) Plot and label each of the following on the grid:
   (1; 1); (–2, 2);
   (0; 2); (–1; 3);
   (0; 0); (–3; 0);
   (2; –1); (–3; –2)



§ 5.10 CALCULATION OF x AND y VALUES

✧ To plot graphs, points have to be calculated from a given equation. We choose suitable values for the x-coordinate, substitute these into the equation, and calculate the y-coordinate.

EXAMPLES	SOLUTIONS
1) Given $y = 2x - 1$ a) Calculate the y values for: i) $x = -3$ ii) $x = 0$ iii) $x = 1$ b) Write down the coordinates of each point c) Write the coordinates in table form	1) a) i) $y = 2(-3) - 1 = -6 - 1 = -7$ ii) $y = 2(0) - 1 = 0 - 1 = -1$ iii) $y = 2(1) - 1 = 2 - 1 = 1$ b) $(-3; -7)$, $(0; -1)$, $(1; 1)$ c) **x**: -3, 0, 1 **y**: -7, -1, 1
2) Given $y = 3x^2 + 2$. For each of the following x values, calculate the y value: a) $x = 1$ b) $x = 0$ c) $x = -1$	2) a) $y = 3(1)^2 + 2 = 3 + 2 = 5$ b) $y = 3(0)^2 + 2 = 0 + 2 = 2$ c) $y = 3(-1)^2 + 2 = 3 + 2 = 5$
3) $y = \frac{8}{x}$ Calculate the values of y for: a) $x = -2$ b) $x = 4$	3) a) $y = \frac{8}{-2} = -4$ b) $y = \frac{8}{4} = 2$

Exercise 5.10

1) Given $y = 3x + 1$.

a) Calculate the y values for following x values:

i) $x = -3$ ....

ii) $x = 0$ ....

iii) $x = 2$ ....

b) Write the coordinates in table form

x	-3	0	2
y			

c) Write down the coordinates of each point

....

2) Given that $y = 2x^2 - 1$. For each x value, calculate the corresponding y value:

a) $x = -3$ ....

b) $x = 0$ ....

c) $x = 2$ ....

3) Given $y = \frac{4}{x}$. Calculate the y values for the following x values:

a) $x = 4$ ....

b) $x = -2$ ....

§ 5.11 FUNCTION NOTATION

✧ Instead of $y = 2x + 5$, we sometimes write $f(x) = 2x + 5$. $f(x)$ is read "f of x". This notation is useful when finding the coordinates of points which lie on a graph.

EXAMPLE	SOLUTION
You are given $f(x) = 2x + 5$.	
1) Calculate a) $f(2)$ b) $f(-1)$ c) $f(0)$	1) a) $f(2) = 2(2) + 5 = 4 + 5 = 9$ b) $f(-1) = 2(-1) + 5 = -2 + 5 = 3$ c) $f(0) = 2(0) + 5 = 0 + 5 = 5$
2) Use your calculations to write down the coordinates of four points on the graph of $f(x) = 2x + 5$	2) Points on the graph are: (2; 9), (– 1; 3), (0; 5)
3) Calculate the value of x if: a) $f(x) = 7$	3) a) $2x + 5 = 7$ $2x + 5 - 5 = 7 - 5$ $2x = 2$ $\frac{2x}{2} = \frac{2}{2}$ $x = 1$
b) $f(x) = -3$	b) $2x + 5 = -3$ $2x + 5 - 5 = -3 - 5$ $2x = -8$ $\frac{2x}{2} = \frac{-8}{2}$ $x = -4$

Exercise 5.11

If $f(x) = 6 - 4x$,

1) Calculate:

a) $f(0)$ ....

b) $f(-1)$ ....

c) $f(-2)$ ....

d) $f(3)$ ....

2) Use your calculations to write down the coordinates of four points on the graph of f

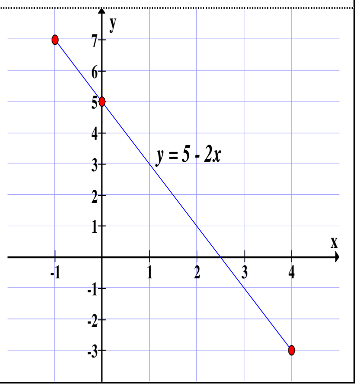
....

3) Calculate x if:

a) $f(x) = 26$ b) $f(x) = 5$ c) $f(x) = -6$

§ 5.12 DRAWING THE GRAPH OF A STRAIGHT LINE FROM THE EQUATION

✧ You can draw the graph of a straight line from its equation by using the equation to find the co-ordinates of 3 points on a line.

EXAMPLES	SOLUTIONS
1) Use a table of values for $x = -1$, 0 and 4 to work out the co-ordinates of 3 points on the graph of $y = 5 - 2x$	When $x = -1$, $y = 5 - 2(-1) = 5 + 2 = 7$ When $x = 0$, $y = 5 - 2(0) = 5 - 0 = 5$ When $x = 4$, $y = 5 - 2(4) = 5 - 8 = -3$ So the table of values is: x: -1, 0, 4 y: 7, 5, -3
2) Use these points to draw the graph of $y = 5 - 2x$	Procedure • draw axes • number the x axis from -1 to 4 • number the y axis from -3 to 7 • plot the points from the table • check that they lie on a straight line • draw the straight line with a ruler • label the line with its equation 

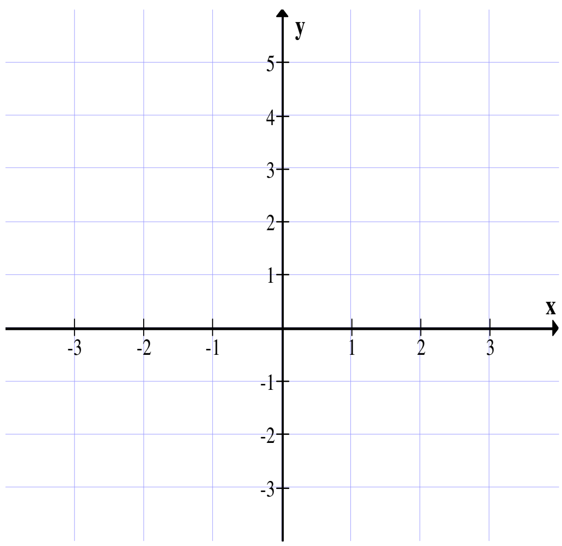
Exercise 5.12

1) Use $x = -1$; $x = 0$ and $x = 1$ to work out the co-ordinates of 3 points on the following graphs: $y = 3x$, $y = -2x$ and $y = x + 4$.

	$y = 3x$	$y = -2x$	$y = x + 4$
$x = -1$	$y = 3(-1) = -3$		
$x = 0$			
$x = 1$			

2) Use these points to draw the graphs of $y = 3x$, $y = -2x$ and $y = x + 4$ on the given set of axes.

3) Label each of the graphs with its equation.



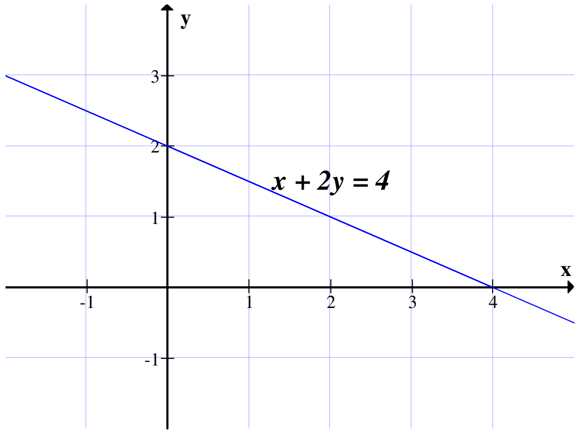
§ 5.13 DRAWING THE GRAPH OF A STRAIGHT LINE BY FINDING THE x- AND y- INTERCEPT

- Some equations of a straight line, such as $\boldsymbol{x + 2y = 4}$ are not given as "y = ...".
- Two easy points to find on such a graph are usually where $\boldsymbol{x = 0}$ and $\boldsymbol{y = 0}$. These are the points where the line cuts the x - and y - axes. (Remember: **axis** is singular and **axes** are plural).

When $x = 0$, $(0) + 2y = 4$
$2y = 4$
$\therefore \quad y = 2$
So the point **(0; 2)** lies on the line

When $y = 0$, $x + 2(0) = 4$
$x + 0 = 4$
$\therefore \quad x = 4$
So the point **(4; 0)** lies on the line

We plot the points (0; 2) and (4; 0), join them up with a straight line and label the graph.



Exercise 5.13

1) By finding the points where $x = 0$ and $y = 0$, draw and label the graphs of the following straight lines on the same set of axesa) $x + y = 5$ b) $x + y = 3$

a) $x + y = 5$

When $x = 0$, $(0) + y = 5$

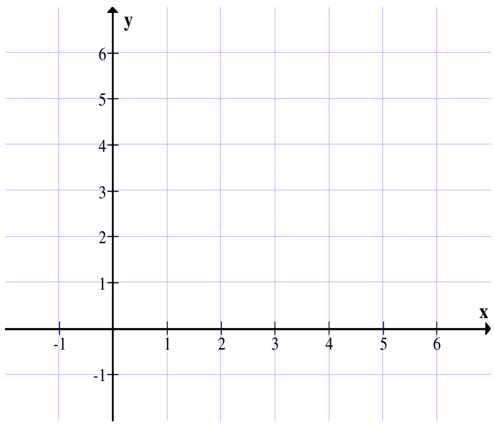
....

So the point .... lies on the graph

When $y = 0$, $x + (0) = 5$

....

So the point .... lies on the graph



b) $x+ y = 3$

When $x = 0$, ... So the point .... lies on the graph

When $y = 0$, ... So the point .... lies on the graph

2) Write down what you notice about the two graphs:

……………………………………………………………………………………………………

……………………………………………………………………………………………………

……………………………………………………………………………………………………

§ 5.14 MORE STRAIGHT LINE GRAPHS

✧ When the axes are not given, you will have to draw the x- and y- axes at right angles to one another, and then plot the values of x and y on the axes

EXAMPLE

Draw the graph of $x + 3y = 6$ by finding the cut on the x-axis and the cut on the y-axis.

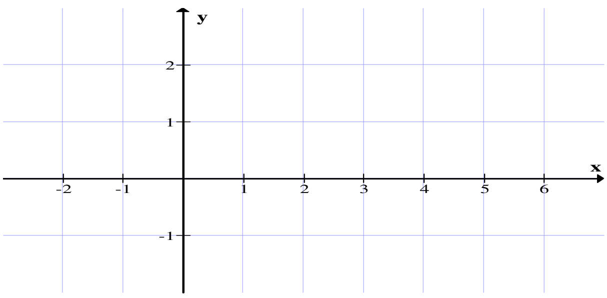
SOLUTION

Step 1: Substitute $x = 0$ and $y = 0$ in the equation

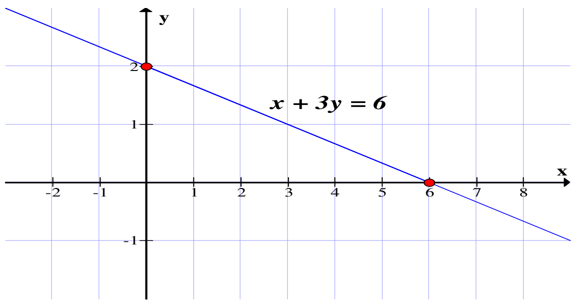
When $x = 0$, $(0) + 3y = 6$
$3y = 6$
$y = 2$
So the point (0; 2) lies on the graph

When $y = 0$, $x + 3(0) = 6$
$x + 0 = 6$
$x = 6$
So the point (6; 0) lies on the graph

Step 2:
Draw in the axes and label them. Make sure that you can plot $x = 6$ and $y = 2$ on your axes.



Step 3:
Plot the two points, join them up and label the line.



<u>Exercise 5.14</u>

Draw the following graphs by finding the cuts on the x-axis and the cut on the y-axis.
Don't forget to label each graph.

1) $x + 2y = 2$

When $x = 0$, ...

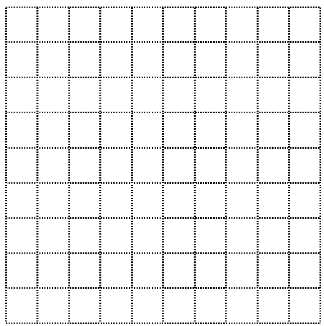
....

So the point .... lies on the graph

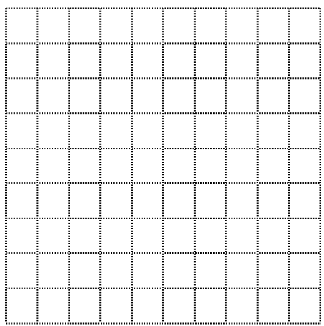
When $y = 0$, ...

....

So the point .... lies on the graph



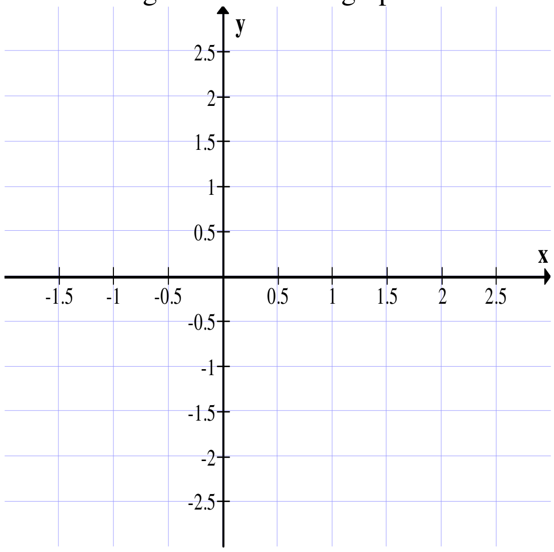
2) $3x - 2y = 6$



§ 5.15 INVESTIGATING CHARACTERISTICS OF STRAIGHT LINES

Exercise 5.15

1a) Draw the following three graphs on the same set of axes by finding the cuts on the *x*-axis and cuts on the *y*-axis: $y = 2x + 1$; $y = 2x - 2$ and $y = 2x - 1$. Don't forget to label each graph.



1b) Study the three graphs and then write down

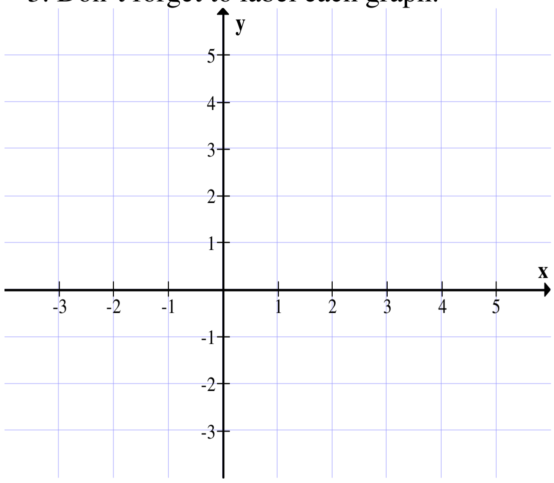
i) What is the same?

…………………………………………………………………………………………………

ii) What is different?

…………………………………………………………………………………………………..

2a) Draw the following three graphs on the same set of axes by finding the cuts on the *x*-axis and cuts on the *y*-axis: $y = -x + 5$; $y = -x + 1$ and $y = -x - 3$. Don't forget to label each graph.



2b) Study the three graphs and then write down

i) What is the same?

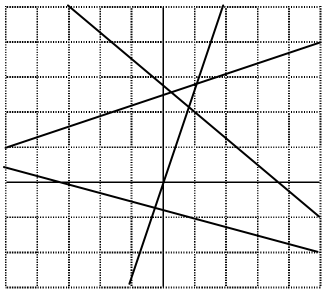
…………………………………………………………………………………………………

ii) What is different?

…………………………………………………………………………………………………

§ 5.16 THE SLOPE OF A LINE

- These straight lines slope in **different directions**. Some have steeper slopes than others. We use the word **gradient** to describe the slope of a line.
- You read words from *left to right*. We read the slope of a line in the same way.

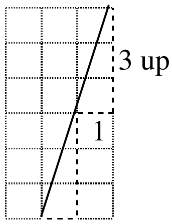


EXAMPLE 1

This straight line slopes **up**.

It goes *up 3 units for each 1 unit across*.

We say its gradient is 3.



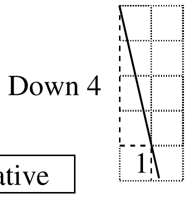
EXAMPLE 2

This straight line slopes **down**.

It goes *down 4 units for each 1 unit across*.

We say its gradient is – 4.

Remember: up ↑ is positive; down ↓ is negative

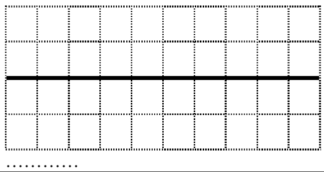


EXAMPLE 3

This straight line is **flat**.

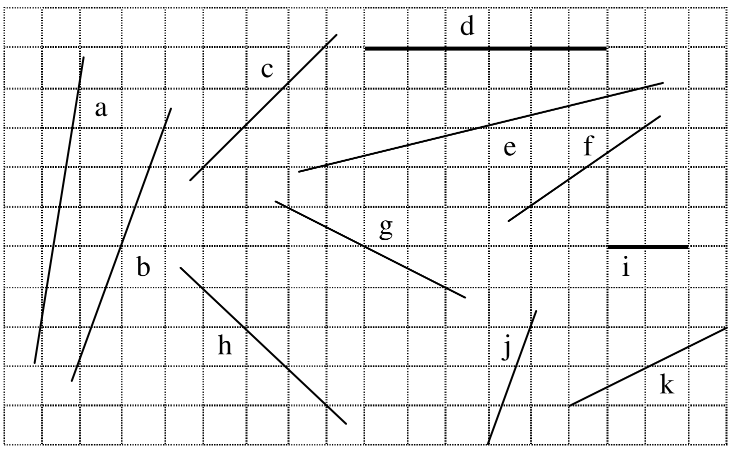
It goes neither up nor down.

It has no slope, so its gradient is zero.



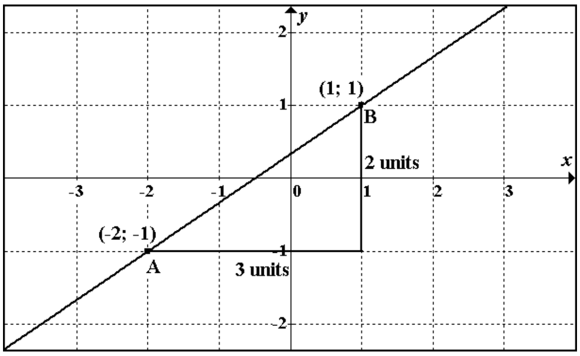
Exercise 5.16

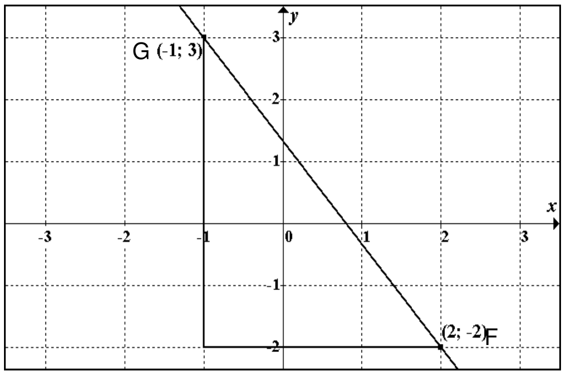
Find the gradients of each of the lines on the grid



§ 5.17 WORKING OUT THE GRADIENT OF A STRAIGHT LINE

- To move from A to B, move 2 units **up** (**positive** direction) and 3 units to the **right** (**positive** direction).
- **Gradient** of AB = $\frac{\text{difference in y values}}{\text{difference in x values}} = \frac{2}{3}$
- Any two points on the line may be chosen to calculate the gradient



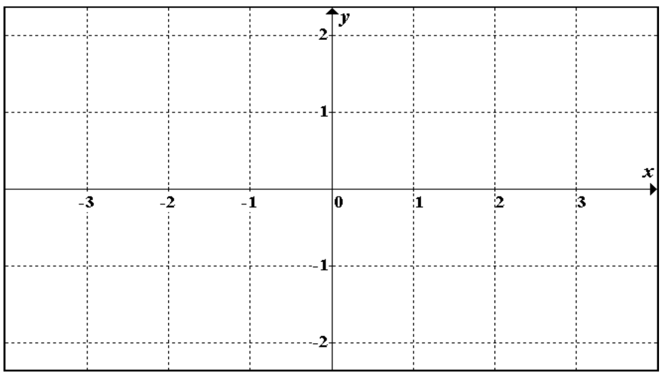
EXAMPLES	SOLUTIONS
1) On the given grid, draw the straight line graph that passes through the points F(2; –2) and G(–1; 3)	
2) Find the gradient of the line joining these points	From F, move 3 units **left** (**negative** direction) and 5 units **up** (**positive** direction) to G. Gradient of FG = $\frac{\text{difference in y values}}{\text{difference in x values}}$ $= \frac{5}{-3} = -\frac{5}{3}$

Exercise 5.17

1) On the given grid, draw a straight line graph that passes through the points M(–2; 1) and N(1; –1)

2) Find the gradient of the line joining M and N.

   Gradient of MN = ...

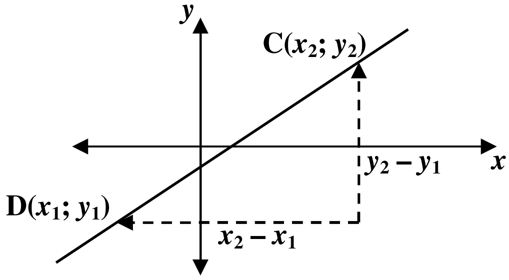


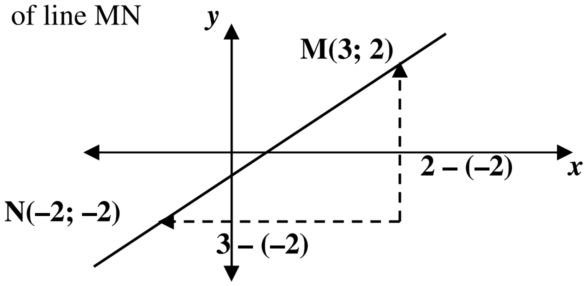
§ 5.18 THE GRADIENT FORMULA

✧ Using the coordinates of points C and D, we obtain the formula:

$$\text{Gradient of CD} = \frac{\text{difference in y values}}{\text{difference in x values}}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

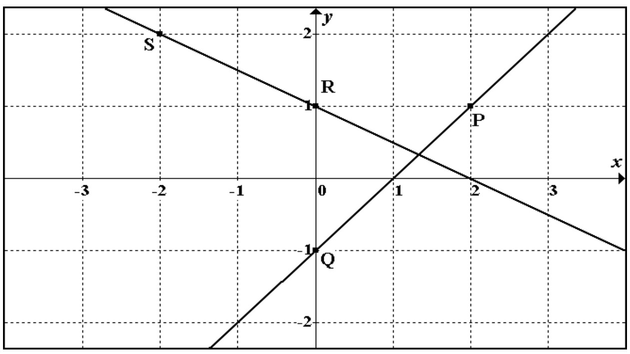


EXAMPLES	SOLUTIONS
1) Use the following diagram to find the gradient of line MN 	Gradient of MN = $\frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{2-(-2)}{3-(-2)}$ $= \frac{2+2}{3+2} = \frac{4}{5}$
2) Calculate the gradient of the line which passes through the points K(1; –4) and L(–3, –2). **<u>Note:</u>** In the formula, it does not matter which point you start with, as long as you start with the **<u>same</u>** point for both the x and y coordinates.	Gradient of KL = $\frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{-4-(-2)}{1-(-3)}$ or $\frac{-2-(-4)}{-3-1}$ $= \frac{-4+2}{1+3}$ or $\frac{-2+4}{-4}$ $= -\frac{1}{2}$

<u>Exercise 5.18</u>

1a) Write down the coordinates of points P, Q, R, and S.

P(;) Q(;) R(;) S(;)



1b) Use these points and the formula to find gradient of PQ and gradient of RS

Gradient of PQ = ...

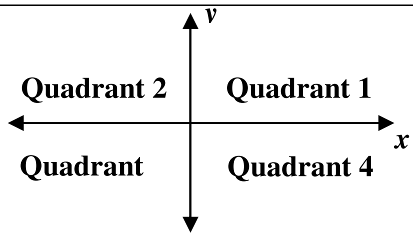
Gradient of RS = ...

2) Without the use of a diagram, find the gradient of the line GH that passes through the points G(3; –2) and H(–1; 4)

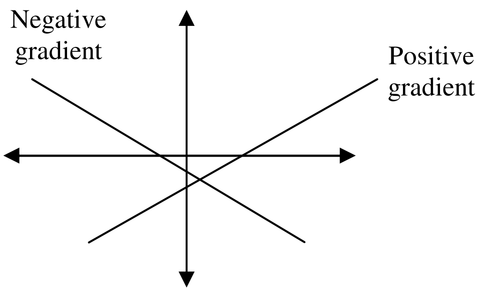
Gradient of GH = ...

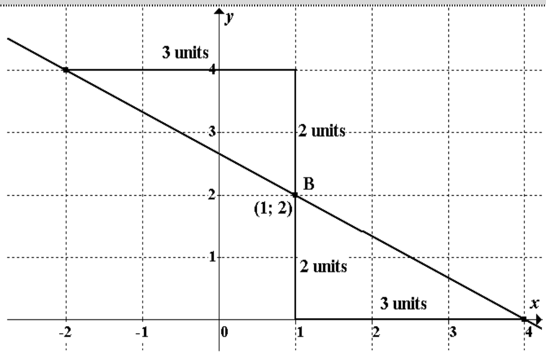
§ 5.19 USING THE GRADIENT

- The x and y axes divide the grid into **four quadrants** which are numbered in the fixed order shown.



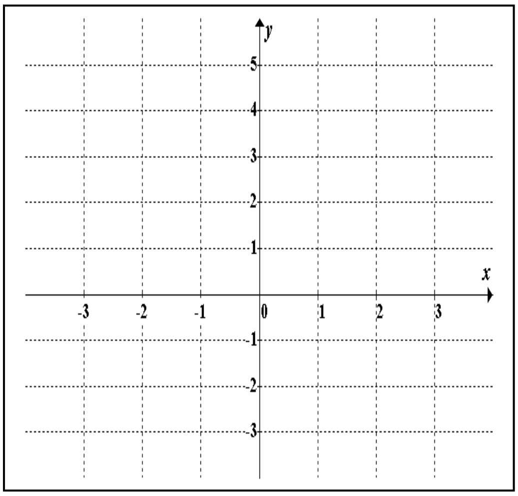
- The lines that slope **up** from left to right have a **positive** gradient.
- The lines that slope **down** from left to right have a **negative** gradient.
- From the diagram, you can see the direction in which lines having a positive or a negative gradient will run.



EXAMPLE	SOLUTION
Use the grid to draw the line that passes through the point B(1; 2) and which has a gradient of $-\frac{2}{3}$	 Gradient of AB $= -\frac{2}{3} = \frac{-2}{3} = \frac{2}{-3}$ From B(1; 2) move: 2 units down and 3 units to the right, OR 2 units up and 3 units to the left, and then join the points

<u>Exercise 5.19</u>

Use the grid to draw the line that passes through the point C (–1; 1) and which has a gradient of $-\frac{1}{4}$.



1) Write down the coordinates of two other points which lie on this graph.

……………………..............……........................

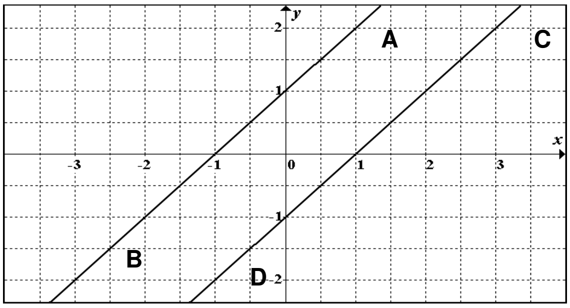
2) Is the gradient positive or negative?

…………………………....................................

3) Between which quadrants does this line run?

……………..……..

§ 5.20 GRADIENTS OF PARALLEL AND PERPENDICULAR LINES



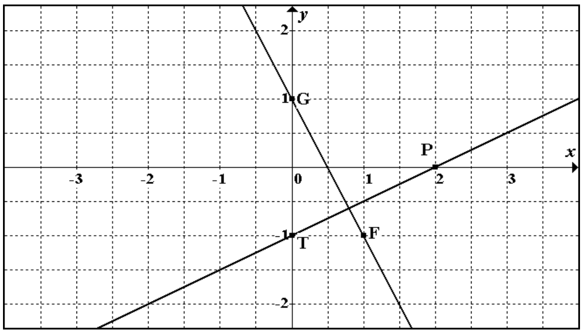
The lines AB and CD are **parallel**.

Gradient of AB = ……………………….......

Gradient of CD = …………..……….........

What do you notice? ....

......……………………………………….......



The lines FG and PT are p**erpendicular**.

Gradient of FG = …………………………......

Gradient of PT = …………………………......

Gradient of FG × Gradient of PT= …………..

What do you notice? ....

......……………………………………….......

Note:

The product of the gradients of perpendicular lines is – 1

EXAMPLE	SOLUTION
AB has a gradient of $\frac{3}{4}$. 1) Find the gradient of CD, which is parallel to AB	Gradient of CD = $\frac{3}{4}$
2) Find the gradient of FG, which is perpendicular to AB	Gradient of FG= $-\frac{4}{3}$, since $\frac{3}{4} \times -\frac{4}{3} = -1$

Exercise 5.20

1) Decide whether lines AB and CD are parallel, perpendicular, or neither:

	Gradient of AB	Gradient of CD	Working	AB ∥ CD or AB ⊥ CD or neither
a)	-2	$\frac{1}{2}$		
b)	-3	$\frac{1}{3}$		
c)	2,5	$2\frac{1}{2}$		
d)	$-1\frac{1}{2}$	$\frac{2}{3}$		

2) Find the gradient of the line perpendicular to the line having gradient a) $\frac{3}{4}$ b) -5 c) $-2\frac{1}{3}$

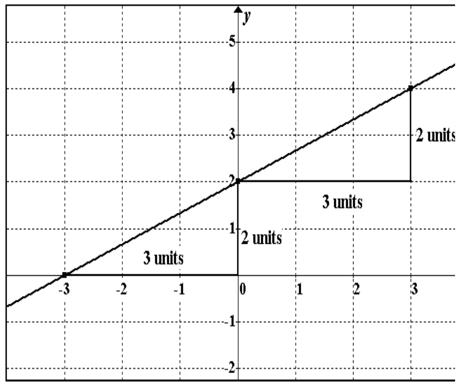
a) .... b) .... c) ....

§ 5.21 DRAWING STRAIGHT LINE GRAPHS USING THE GRADIENT-INTERCEPT METHOD

✧ When an equation of a straight line is written in the form $\boldsymbol{y = ax + q}$, then
- $\boldsymbol{q}$ is **the y-intercept** and
- $\boldsymbol{a}$ is **the gradient**.

STEPS TO FOLLOW WHEN DRAWING A STRAIGHT LINE GRAPH

Draw the graph of $y = \frac{2}{3}x + 2$



STEP 1: Write down the y-intercept

y-intercept = $q = 2$

STEP 2: Write down the gradient

Gradient = $\frac{2}{3} = \frac{\textit{difference in } y}{\textit{difference in } x} = \frac{-2}{-3}$

STEP 3: Plot the y-intercept of 2 on the y-axis. Then use the gradient to find another point on the graph

- For the gradient of $\frac{2}{3}$, move 3 units right and 2 units up from the y-intercept of 2.
- Or, for the gradient of $\frac{-2}{-3}$, move 2 units down and 3 units left from the y-intercept of 2.

<u>Note:</u>

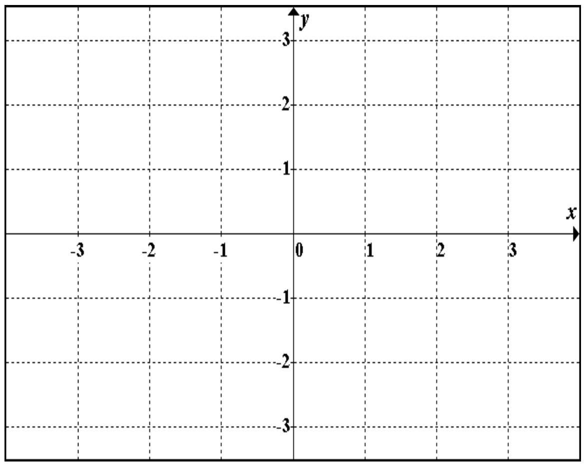
- $y = 2x$ means $y = 2x + 0$, so the y-intercept is zero
- $y = \frac{2}{3}x$ can also be written $y = \frac{2x}{3}$

<u>Exercise 5.21</u>

1) Complete the table for the four given straight line graphs

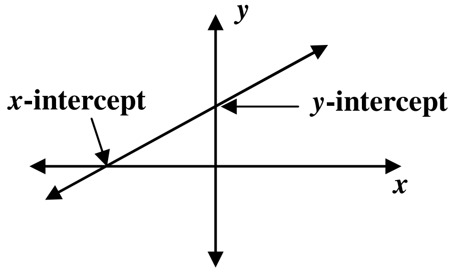
	a	q
a) $y = 2x + 1$		
b) $y = -x + 2$		
c) $y = x$		
d) $y = \frac{x}{2} - 1$		

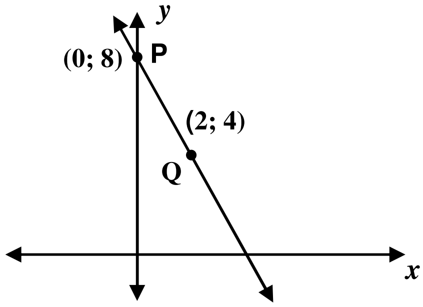
2) Use the gradient-intercept method to plot the four graphs. Don't forget to label them.



§ 5.22 FINDING THE EQUATION OF A STRAIGHT LINE GIVEN THE y-INTERCEPT AND THE CO-ORDINATES OF 1 OTHER POINT

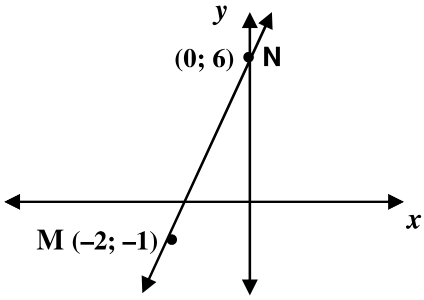
- The point where the graph cuts the ***x*-axis** is the ***x*-intercept.** At this point the y co-ordinate is zero.
- The point where the graph cuts the ***y*-axis** is the ***y*-intercept.** At this point, the ***x*** co-ordinate is zero.



EXAMPLE	SOLUTION
P(0; 8) and Q(2; 4) are points on the line PQ. Find the equation of line PQ. 	P (0; 8) is the y-intercept, so we know that $q = 8$ The gradient of PQ = $\frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{8-4}{0-2}$ $= \frac{4}{-2}$ $= -2$ This means that $a = -2$ The general form of the equation is $y = ax + q$ $\therefore$ The equation of PQ is $y = -2x + 8$

Exercise 5.22

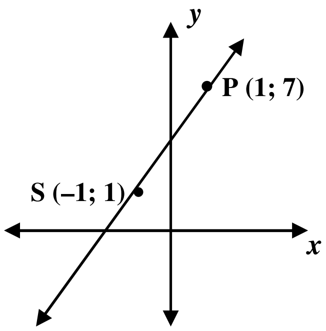
1) Find the equation of the line through the point M(–2; –1) which cuts the *y*-axis at N(0; 6).



2) Points A(2; –3) and B(0; –1) lie on a line. Find the equation of line AB.

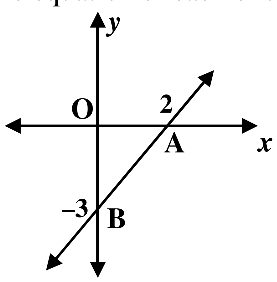
§ 5.23 FINDING THE EQUATION OF A STRAIGHT LINE GIVEN THE CO-ORDINATES OF 2 POINTS ON THE LINE

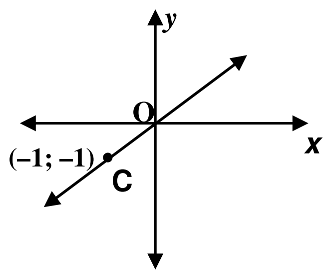
- In the previous example, we found the equation of a straight line given the *y*-intercept and one point.
- Sometimes you will not be given the *y*-intercept but another point on the line.

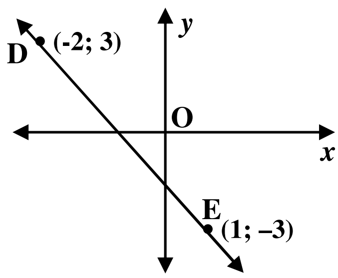
	EXAMPLE	SOLUTION
1)	Find the equation of the line having a gradient of – 9 and passing through the point (0; – 4).	The gradient is – 9, so $a = -9$ The point (0; –4) is the *y*-intercept, so $q = -4$. The general form of the equation is $y = ax + q$ So the equation of this line is $y = -9x - 4$
2)	Points S(– 1; 1) and P(1; 7) lie on the line in the sketch. Find the equation of the line. 	Gradient of SP = $\dfrac{y_2 - y_1}{x_2 - x_1}$ $= \dfrac{7-1}{1-(-1)} = \dfrac{6}{2} = 3$ So $a = 3$ Substitute into the general form of the equation $y = ax + q$ We get $y = 3x + q$ Substitute S (–1; 1) $1 = 3(-1) + q$ $1 = -3 + q$ $4 = q$ OR substitute P(1; 7): $7 = 3(1) + q$ $7 = 3 + q$ $4 = q$ So the equation of line PS is $y = 3x + 4$ **Note:** • Either of the two given points may be used for substitution

Exercise 5.23

Find the equation of each of the following lines:

1) 

2) 

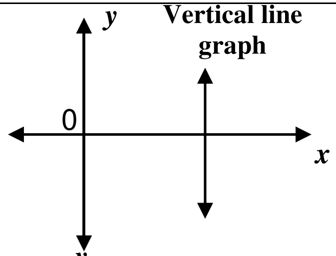
3) 

§ 5.24 FINDING THE EQUATION OF VERTICAL AND HORIZONTAL LINES

- Along a **<u>vertical line</u>:** the ***y* values change** but the ***x* value remains constant**. So $x_2 - x_1 = 0$.
- Gradient of a vertical line

$$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{0} = \text{undefined.}$$

- Remember that division by zero is undefined.

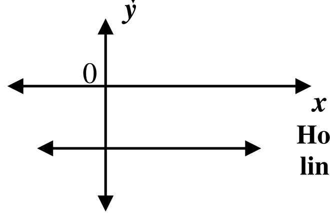


Vertical line graph

- Along a **<u>horizontal line</u>:** the ***x* values change**, but the ***y* value remains constant.** So $y_2 - y_1 = 0$.
- Gradient of a horizontal line

$$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{0}{x_2 - x_1} = 0.$$

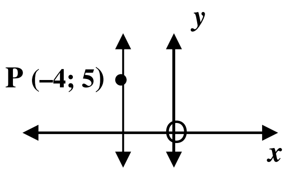
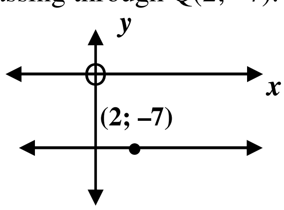
- Remember that zero divided by any non-zero number is zero



Horizontal line graph

<u>Note:</u>

- the gradient of any **vertical line** is **undefined**
- the gradient of any **horizontal line** is **zero** (there is no slope)
- Because the gradient of a **vertical line** is undefined, we cannot use the equation formula

EXAMPLES	SOLUTIONS
1) Find the equation of the vertical line passing through P(– 4; 5). 	Each point on the line has an x co-ordinate of –4, but each point has a different y co-ordinate. The vertical line has equation $x = -4$.
2) Find the equation of the horizontal line passing through Q(2; –7). 	Each point on the line has a y-coordinate of –7 and a different x-coordinate. The horizontal line has equation $y = -7$.

<u>Exercise 5.24</u>

1) a) What is the equation of the vertical line through A(–2; –1)?

……………………………………………………………

b) What is the gradient of this line? ……………………………

2) a) What is the equation of the horizontal line through B (– 5; 3)?

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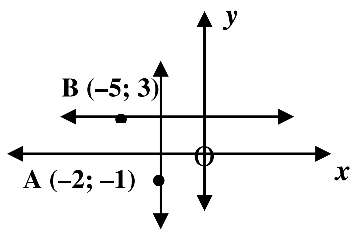
b) What is the gradient of this line? ……………………………



CHAPTER 6
Quadratic equations and functions Hyperbolic functions, and Exponential equations and functions

In this chapter you will

- Solve quadratics equations with 2 or 3 terms
- Solve exponential equations using a table, and algebraically
- Draw quadratic functions
- Study the effect of 'a' on the drawing of both $y = ax^2$ and $y = ax^2 + q$.
- Find the cuts on the x-axis of a quadratic function
- Find the equation of a parabola
- Find the intersection between the graph of a parabola and a straight line
- Find the average gradient between 2 points on a parabola
- Draw the graph of a hyperbola
- Study the effect of 'a' and 'q' on the graphs of the hyperbolae $y = \frac{a}{x}$ and $y = \frac{a}{x} + q$.
- Find the intercepts on the x-axis of the hyperbola $y = \frac{a}{x} + q$.
- Find the equation of the hyperbola $y = \frac{a}{x}$ and of the hyperbola $y = \frac{a}{x} + q$
- Draw the exponential graph $y = b^x$, $y = a.b^x$, and $y = a.b^x + q$, $a \neq 0$, $b \neq 0$ and $q \neq 0$
- Find the equation of the exponential function $y = b^x$.

This chapter covers material from Topic 2: Functions

<u>SUBJECT OUTCOME 2.1</u>
Work with a range of functions and patterns and solve problems

<u>Learning Outcome 1:</u> Sketch and interpret quadratic functions centres on the y-axis, $y = ax^2 +$, hyperbolae, $y = \frac{a}{x} + q$, and exponential functions, $y = a.b^x + q$, $b > 0$.

<u>Learning Outcome 2:</u> Convert between numerical, verbal, symbolic and graphical representations of information

<u>Learning Outcome 3:</u> Investigate and generalise the impact of a and q in the functions above.

<u>SUBJECT OUTCOME 2.3:</u>
Solve algebraic equations

<u>Learning Outcome 2:</u> Solve quadratic equations by factorisation

<u>Learning Outcome 3:</u> Solve exponential equations of the form $k.a^{x+p} = m$ *by trial and error*

<u>Learning Outcome 5:</u> Solve equations simultaneously by numerical, algebraic and graphical methods

<u>SUBJECT OUTCOME 2.4</u>
Investigate the average rate of change of a function between two values of the independent variable

<u>Learning Outcome 1</u> Investigate the gradient between two points on a curve with specific reference to linear functions

<u>Learning Outcome 2</u> Investigate the gradient of a curve at a specific point.

§ 6.1 SOLVING QUADRATIC EQUATIONS

A product equal to zero:

- Any number multiplied by zero gives an answer of zero. For example, $3 \times 0 = 0$, $-4 \times 0 = 0$.
- When we multiply two or more numbers and the answer is zero, at least one of the numbers must be zero. For example, if $a \times b = 0$, then $a = 0$ or $b = 0$. We expect two answers.

EXAMPLES

Solve for x and check the answer to the first question:

1) $(x + 3)(x - 2) = 0$ 2) $2x(x - 6) = 0$ 3) $3(x - 1)^2 = 0$

SOLUTIONS

1)

$$(x + 3)(x - 2) = 0$$

$x + 3 = 0$ or $x - 2 = 0$

$x + 3 - 3 = 0 - 3$ $x - 2 + 2 = 0 + 2$

$x = -3$ or $x = 2$

Check:	LHS	RHS
Substitute $x = -3$	$(x + 3)(x - 2)$ $= (3 + 3)(3 - 2)$ $= (0)(5)$ $= 0$	0
LHS = RHS		
Substitute $x = 2$	$(x + 3)(x - 2)$ $= (2 + 3)(2 - 2)$ $= (5)(0)$ $= 0$	0
LHS = RHS		

2)

$$2x(x - 6) = 0$$

$2x = 0$ or $x - 6 = 0$

$\frac{2x}{2} = \frac{0}{2}$ $x - 6 + 6 = 0 + 6$

$x = 0$ or $x = 6$

3)

$$3(x - 1)^2 = 0$$

$$3(x - 1)(x - 1) = 0$$

$$\frac{3(x-1)(x-1)}{3} = \frac{0}{3}$$

$$(x - 1)(x - 1) = 0$$

$$x - 1 = 0$$

$$x = 1$$

Although we expect two answers, we have only one since the two brackets are the same.

Exercise 6.1

Solve the equations for the unknown. Do a check for question 1:

1) $(2 - x)(x + 1) = 0$

2) $4(2 - x)^2 = 0$

3) $(2x + 4)(x - 5) = 0$

4) $5c(2c + 4) = 0$

5) $(2x - 3)(x + 4)(1 - x) = 0$

§ 6.2 SOLVING QUADRATIC EQUATIONS WITH TWO TERMS

EXAMPLES	
Solve for x and check the answer to the first question: 1) $4x^2 - 8x = 0$ 2) $25a^2 = 4$	
SOLUTIONS	
1) $4x^2 - 8x = 0$ *Take out a common factor* $4x(x-2) = 0$ $4x = 0$ or $x - 2 = 0$ $\frac{4x}{4} = 0$ $\quad x - 2 + 2 = 0 + 2$ $x = 0$ $\quad x = 2$	2) $25a^2 = 4$ $25a^2 - 4 = 0$ *This is the difference of two squares* $(5a-2)(5a+2) = 0$ $5a - 2 = 0$ or $5a + 2 = 0$ $5a - 2 + 2 = 0 + 2$ $\quad 5a + 2 - 2 = 0 - 2$ $5a = 2$ $\quad 5a = -2$ $\frac{5a}{5} = \frac{2}{5}$ $\quad a = -\frac{5}{2}$ $a = \frac{2}{5}$

Exercise 6.2

Solve the equations. Do a check for question 1:

1) $x^2 - 4x = 0$

2) $3x^2 = 9x$

3) $3d^2 = 243$

4) $x^2 = 16$

5) $3x^2 - 27 = 0$

§ 6.3 SOLVING QUADRATIC EQUATIONS WITH THREE TERMS

✧ To solve trinomial quadratic equations, first look for a common factor to take out. Then factorise the trinomial and continue as before.

EXAMPLES

Solve:

1) $x^2 - 3x - 4 = 0$ 2) $3x^2 + 2x - 8 = 0$ 3) $12a^2 + a - 35 = 0$ 4) $10a^2 - 5a - 5 = 0$

SOLUTIONS

1)
$$x^2 - 3x - 4 = 0$$
$$(x - 4)(x + 1) = 0$$
$$x - 4 = 0 \quad \text{or} \quad x + 1 = 0$$
$$x - 4 + 4 = 0 \qquad x + 1 - 1 = 0 - 1$$
$$x = 4 \quad \text{or} \quad x = -1$$

2)
$$3x^2 + 2x - 8 = 0$$
$$(3x - 4)(x + 2) = 0$$
$$3x - 4 = 0 \quad \text{or} \quad x + 2 = 0$$
$$3x - 4 + 4 = 0 + 4 \qquad x + 2 - 2 = 0 - 2$$
$$3x = 4 \qquad x = -2$$
$$x = \frac{4}{3}$$

3)
$$12a^2 + a - 35 = 0$$
$$(4a + 7)(3a - 5) = 0$$
$$4a + 7 = 0 \quad \text{or} \quad 3a - 5 = 0$$
$$4a + 7 - 7 = 0 - 7 \qquad 3a - 5 + 5 = 0 + 5$$
$$4a = -7 \qquad 3a = 5$$
$$\frac{4a}{4} = \frac{-7}{4} \qquad \frac{3a}{3} = \frac{5}{3}$$
$$a = \frac{-7}{4} \qquad a = \frac{5}{3}$$

4)
$$10a^2 - 5a - 5 = 0$$
$$5(2a^2 - a - 1) = 0$$
$$5(2a + 1)(a - 1) = 0$$
$$2a + 1 = 0 \quad \text{or} \quad a - 1 = 0$$
$$2a + 1 - 1 = 0 - 1 \qquad a - 1 + 1 = 0 + 1$$
$$2a = -1 \qquad a = 1$$
$$\frac{2a}{2} = \frac{-1}{2}$$
$$a = -\frac{1}{2}$$

Exercise 6.3

Solve:

1) $x^2 + 2x + 1 = 0$

2) $x^2 + x - 2 = 0$

3) $x^2 - 3x + 2 = 0$

4) $2x^2 - 5x - 3 = 0$

5) $2a^2 - 10a + 12 = 0$

6) $3x^2 + 5x - 2 = 0$

§ 6.4 SOLVING EXPONENTAIL EQUATIONS USING A TABLE

✧ Exponential equations are of the form $a^x = 0$. The unknown is the exponent.

Exercise 6.4

1) Complete the table below:

x	–4	–3	–2	–1	0	1	2	3	4
2^x								8	
3^x					1				81
4^x		$\frac{1}{64}$							256

EXAMPLE
Use the completed table to find x in each of the equations:
1) $2^x = 8$ 2) $4^x = 64$ 3) $3^x = 81$ 4) $4^x = \frac{1}{64}$ 5) $2.3^x = 2$

SOLUTION				
1) $2^x = 8$ $x = 3$	2) $4^x = 64$ $x = 3$	3) $3^x = 81$ $x = 4$	4) $4^x = \frac{1}{64}$ $x = -3$	5) $2.3^x = 2$ $3^x = \frac{2}{2} = 1$ $x = 0$

2) Use the completed table to find x in each of the equations:

a) $4^x = 1$ b) $3^x = \frac{1}{9}$ c) $2^x = 2$

d) $3^x = \frac{1}{27}$ e) $4^x = \frac{1}{16}$ f) $2^x = \frac{1}{8}$

g) $4.2^x = 4$ h) $2.3^x = 162$ i) $12.3^x = 12$

j) $2^x = 3^x$ k) $4^x = 4$ l) $15.3^x = 45$

m) $4^{2x} = 256$ n) $3^{3x} = 81$ o) $2^{4x} = 16$

§ 6.5 SOLVING EXPONENTAIL EQUATIONS ALGEBRAICALLY

✧ To solve exponential equations exponentially, we use the fact that if bases are the same, we can equate the exponents.

EXAMPLE				
Solve the following equations algebraically:				
1) $2^x = 8$	2) $4^x = 64$	3) $3^x = 81$	4) $4^x = \frac{1}{64}$	5) $2.3^x = 2$
SOLUTION				
1) $2^x = 8$ $2^x = 2^3$ *Since the bases are the same, we can equate the exponents* $x = 3$	2) $4^x = 64$ $4^x = 4^3$ *Since bases are the same,* $x = 3$	3) $3^x = 81$ $3^x = 3^4$ *Since bases are the same,* $x = 4$	4) $4^x = \frac{1}{64}$ $4^x = 4^{\,3}$ *Since bases are the same,* $x = -3$	5) $2.3^x = 2$ $3^x = \frac{2}{2} = 1$ $3^x = 3^0$ *Since the bases are the same,* $x = 0$

Exercise 6.5

Solve the following exponential equations algebraically:

1) $4^x = 1$
2) $3^x = \frac{1}{9}$
3) $2^x = 2$
4) $3^x = \frac{1}{27}$
5) $4^x = \frac{1}{16}$
6) $2^x = \frac{1}{8}$
7) $4.2^x = 4$
8) $2.3^x = 162$
9) $12.3^x = 12$
10) $2^x = 3^x$
11) $4^x = 4$
12) $15.3^x = 45$
13) $4^{2x} = 256$
14) $3^{3x} = 81$
15) $2^{4x} = 16$

§ 6.6 SOLVING MORE COMPLICATED EXPONENTIAL EQUATIONS

EXAMPLE	
Solve: 1) $4^{x+3} = 64$ 2) $2.3^{x+1} = 18$ 3) $2.4^{x+2} = 8^{-x}$ 4) $3(5^{2x-1}) = 375$	
SOLUTION	
1) $4^{x+3} = 64$ $4^{x+3} = 4^3$ *Since bases are the same* $x + 3 = 3$ $x + 3 - 3 = 3 - 3$ $x = 0$	2) $2.3^{x+1} = 18$ $\frac{2.3^{x+1}}{2} = \frac{18}{2}$ $3^{x+1} = 9$ $3^{x+1} = 3^2$ *Since the bases are the same* $x + 1 = 2$ $x + 1 - 1 = 2 - 1$ $x = 1$
3) $2.4^{x+2} = 8^{-x}$ $2.(2^2)^{x+2} = (2^3)^{-x}$ $2^1.2^{2x+4} = 2^{-3x}$ $2^{2x+5} = 2^{-3x}$ *Since the bases are the same* $2x + 5 = -3x$ $5x = -5$ $\frac{5x}{5} = \frac{-5}{5}$ $x = -1$	4) $3(5^{2x-1}) = 375$ $\frac{3(5^{2x-1})}{3} = \frac{375}{3}$ $5^{2x-1} = 125$ $5^{2x-1} = 5^3$ *Since the bases are the same* $2x - 1 = 3$ $2x = 4$ $x = 2$

Exercise 6.6

Solve the following equations for x:

1) $3^{x-1} = 81$

2) $\left(\frac{1}{3}\right)^x = 81$

3) $3.2^{x+1} = 96$

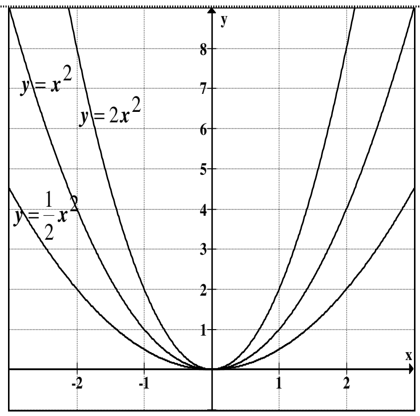
4) $4(3^{2x}) = 324$

5) $5.6^{3x-2} = 180$

6) $\left(\frac{1}{4}\right)^x = 32$

§ 6.7 DRAWING QUADRATIC FUNCTIONS

- A quadratic function has the form $y = ax^2 + q$, and its graph is a **parabola**.
- The simplest form of the equation of a parabola is $y = ax^2$.

EXAMPLE	SOLUTION
1) Draw up a table of values for the following functions: $y = \frac{1}{2}x^2$, $y = x^2$, $y = 2x^2$ where x has values –2; –1, 0; 1; 2	(see table below)
2) Use the table to draw these graphs on the same set of axes	
3) What effect does the value of a have on the graph? How?	a affects the width of the graph – the smaller the value of a, the wider is the graph
4) Write down the turning points of these graphs	Turning points are all (0; 0)

	– 2	– 1	0	1	2
$y = \frac{1}{2}x^2$	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2
$y = x^2$	4	1	0	1	4
$y = 2x^2$	8	2	0	2	8

Exercise 6.7

1) Complete the following table:

x	– 2	– 1	0	1	2
$y = 3x^2$					
$y = \frac{1}{4}x^2$					

2) Use the values in the table to draw the graphs of $y = 3x^2$ and $y = \frac{1}{4}x^2$, and label each graph.

3) Give the value of each one's turning point.

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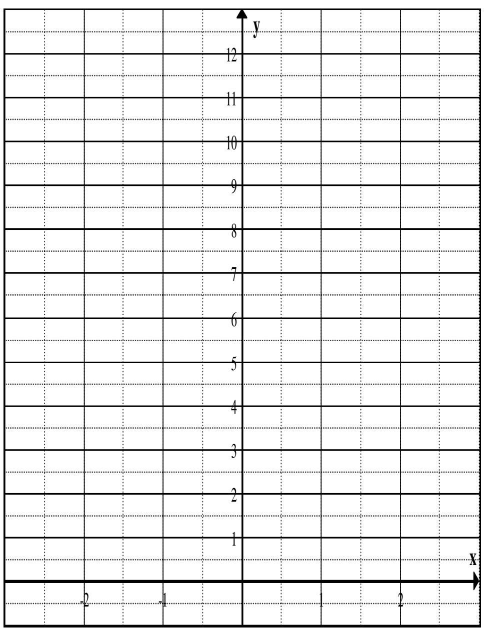
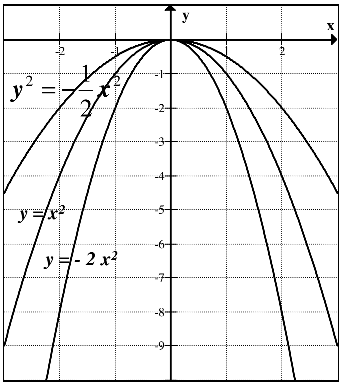


§ 6.8 THE EFFECT OF "a" ON THE DRAWING OF $y = ax^2$

EXAMPLES	SOLUTIONS
1) Draw up a table of values for the following functions: $y = -\frac{1}{2}x^2$, $y = -x^2$, and $y = -2x^2$ where x has values –2; –1, 0; 1; 2	see table below
2) Use the table to draw these graphs on the same set of axes	

	– 2	– 1	0	1	2
$y = -\frac{1}{2}x^2$	– 2	$-\frac{1}{2}$	0	$-\frac{1}{2}$	– 2
$y = -x^2$	– 4	– 1	0	– 1	– 4
$y = -2x^2$	– 8	– 2	0	– 2	– 8

Note:
- If $a > 0$, the graph looks like $\cup$ and the y-coordinate of the turning point is a **minimum**
- If $a < 0$, the graph looks like $\cap$ and the y-coordinate of the turning point is a **maximum**

Exercise 6.8

1) Complete the following table:

x	– 2	– 1	0	1	2
$y = -3x^2$					
$y = -\frac{1}{4}x^2$					

2) Use the values in the table to draw the graphs of $y = -3x^2$ and $y = -\frac{1}{4}x^2$, and label each graph.

3) Give the value of each one's turning point.

…………………………………………

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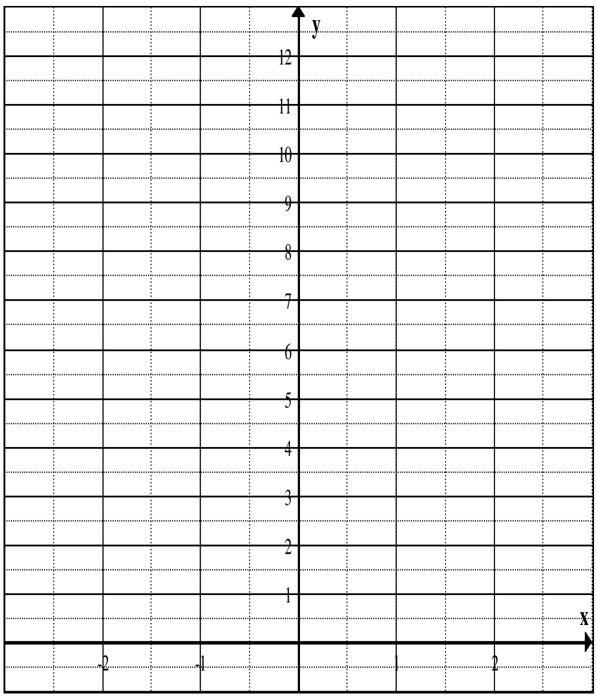
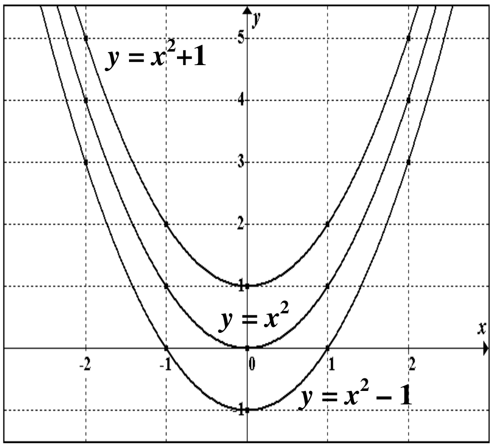


§ 6.9 THE EFFECT OF "q" ON THE DRAWING OF $y = ax^2 + q$

EXAMPLE	SOLUTION
1) Draw up a table of values for the following functions: $y = x^2 + 1$, $y = x^2$, and $y = x^2 - 1$ where x has values –2; –1, 0; 1; 2	see table below
2) Use the table to draw these graphs on the same set of axes	see graph below

	– 2	**– 1**	**0**	**1**	**2**
$\boldsymbol{y = x^2 + 1}$,	5	2	1	2	5
$\boldsymbol{y = x^2}$	4	1	0	1	4
$\boldsymbol{y = x^2 - 1}$	3	0	– 1	0	3



Note:

- The value of q in $y = ax^2 + q$ gives the y-coordinate of the turning point of the parabola.

Equation	$y = x^2 \underline{+ 1}$	$y = x^2 \underline{+ 0}$	$y = x^2 \underline{- 1}$
Turning point	(0; $\underline{1}$)	(0; $\underline{0}$)	(0; $\underline{-1}$)

- The y-axis, $x = 0$, is the axis of symmetry of the graphs

Exercise 6.9

For each of the following equations, give the co-ordinates of the turning point.

1) $y = x^2 - 3$ ……………………………………………………………………………………

2) $y = x^2 + 5$ ……………………………………………………………………………………

3) $y = x^2 - 2$ ……………………………………………………………………………………

4) $y = 2x^2$ ……………………………………………………………………………………

§ 6.10 CUTS ON THE X-AXIS OF A QUADRATIC FUNCTION

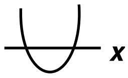
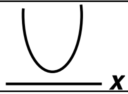
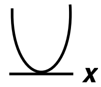
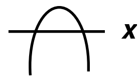
- ✧ At the point where a graph cuts the x-axis, the y-coordinate is zero.
- ✧ The turning point determines whether the graph will cut the x-axis and how many times.

EXAMPLES

For each of the following graphs 1) $y = 2x^2 - 8$ 2) $y = 3x^2 + 9$ 3) $y = x^2$ 4) $y = -x^2 + 1$:

a) Indicate whether a is positive or negative
b) Write down the coordinates of the turning point of each graph
c) Use the turning point to draw a sketch of each graph, showing its shape and x-intercepts
d) From the sketch, write down the number of x-intercepts you expect for each graph
e) Find the x- intercept/s of each graph

SOLUTIONS

1) $y = 2x^2 - 8$	2) $y = 3x^2 + 9$	3) $y = x^2$	4) $y = -x^2 + 1$
a) a positive	a positive	a positive	a negative
b) (0; –8)	(0; 9)	(0; 0)	(0; 1)
c) 			
d) 2	None	1	2
e) $2x^2 - 8 = 0$ $2x^2 = 8$ $2(x^2 - 4) = 0$ $2(x - 2)(x + 2) = 0$ $x = 2$ or $x = -2$	$3x^2 + 9 = 0$ $3x^2 = -9$ $x^2 = -3$ No solution	$x^2 = 0$ $x = 0$	$-x^2 + 1 = 0$ $x^2 - 1 = 0$ $(x - 1)(x + 1) = 0$ $x = 1$ or $x = -1$

Exercise 6.10

1) Indicate whether a is positive or negative
2) Write down the coordinates of the turning point of each graph
3) Use the turning point to draw a sketch of each graph showing its shape and x-intercepts
4) From the sketch, write down the number of x-intercepts you expect for each graph
5) Find the x- intercept/s of each graph:

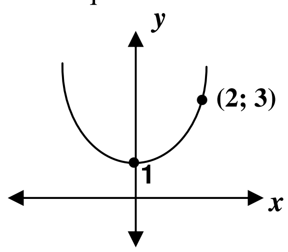
	a) $y = -x^2 + 1$	b) $y = 2x^2 + 1$	c) $y = x^2 - 1$	d) $y = -x^2$
1)				
2)				
3)				
4)				
5)				

§ 6.11 FINDING THE EQUATION OF A PARABOLA

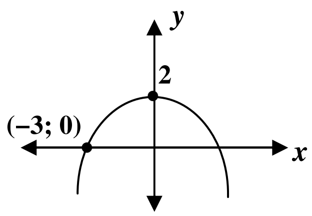
EXAMPLE

Find the equations of each of the parabolas:

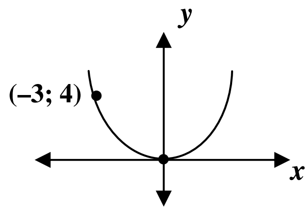
1)



2)



3)



SOLUTION

1) $y = ax^2 + q$

The graph cuts the y-axis at 1, so q = 1

$\therefore \quad y = ax^2 + 1$

$(2; 3) \quad 3 = a(2)^2 + 1$

$3 = 4a + 1$

$2 = 4a$

$\frac{1}{2} = a$

$y = \frac{1}{2}x^2 + 1$

2) $y = ax^2 + q$

The graph cuts the y-axis at 2, so q = 2

$\therefore \quad y = ax^2 + 2$

$(-3; 0) \quad 0 = a(-3)^2 + 2$

$-2 = 9a$

$\frac{-2}{9} = a$

$y = -\frac{-2}{9}x^2 + 2$

3) $y = ax^2 + q$

The graph cuts the y-axis at 0, so q = 0

$\therefore \quad y = ax^2 + 0$

$(-3; 4) \quad 4 = a(-3)^2$

$4 = 9a$

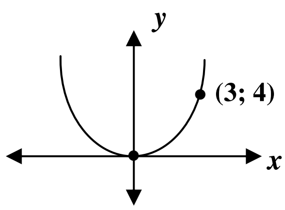
$\frac{4}{9} = a$

$y = \frac{4}{9}x^2$

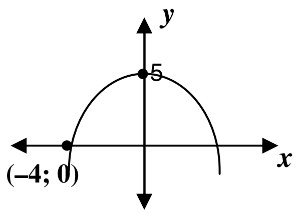
<u>Exercise 6.11</u>

Find the equations of each of the parabolas:

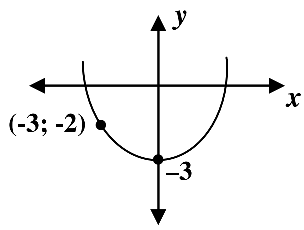
1)



2)



3)



§ 6.12 THE INTERSECTION BETWEEN THE GRAPHS OF A PARABOLA AND A STRAIGHT LINE

EXAMPLE

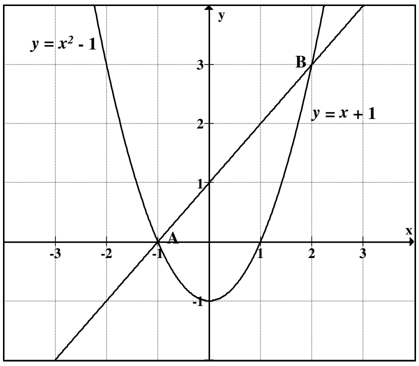
1) Draw up a table of values for a) $y = x^2 - 1$ and b) $y = x + 1$
2) On the same set of axes draw the graphs from the table and label them
3) Write down the coordinates of their points of intersection.

SOLUTION

1)

	– 2	**– 1**	**0**	**1**	**2**
$\boldsymbol{y = x^2 - 1}$	3	0	– 1	0	3
$\boldsymbol{y = x + 1}$	– 1	0	1	2	3

2)



3) Points of intersection:
A (–1; 0); B (2; 3)

Exercise 6.12

1)

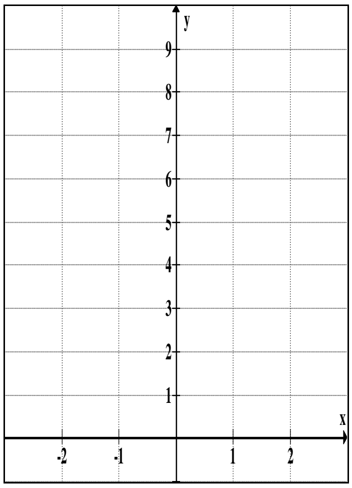
a) Complete the table of values for
i) $y = 2x^2 + 1$
ii) $y = 2x + 5$

	– 2	– 1	0	1	2
$\boldsymbol{y = 2x^2 + 1}$					
$\boldsymbol{y = 2x + 5}$					

b) Draw and label the graphs on the same set of axes

c) Co-ordinates of points of intersection:

....



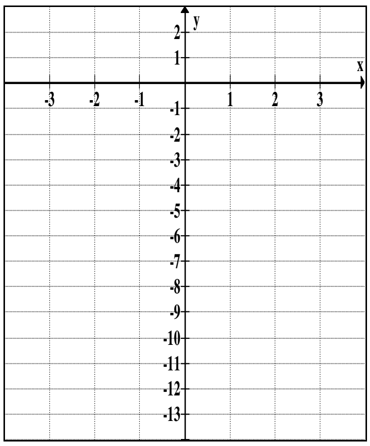
2)

a) Complete the table of values for
i) $y = -x^2 + 2$
ii) $y = 2x - 1$

	– 2	– 1	0	1	2
$\boldsymbol{y = -x^2 + 2}$					
$\boldsymbol{y = 2x - 1}$					

b) Draw and label the graphs on the same set of axes.

c) Co-ordinates of points of intersection: ………………



§ 6.13 FINDING THE AVERAGE GRADIENT BETWEEN TWO POINTS ON A PARABOLA

EXAMPLE

1) Find the average gradient of the parabola p between:

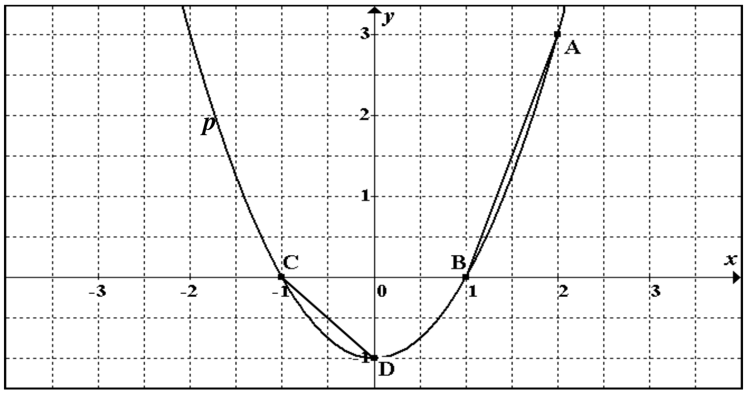
   a) B and A

   b) C and D

2) Is the curve increasing or decreasing between:

   a) A and B?

   b) C and D?



SOLUTION

1) a) A(2; 3) B(1; 0)

Average gradient of p between B and A = Gradient of BA = $\frac{3-0}{2-1} = \frac{3}{1} = 3$.

b) C(–1; 0) D(0; –1)

Average gradient of p between C and D = Gradient of CD = $\frac{0-(-1)}{-1-0} = \frac{0+1}{-1} = -1$.

2) a) The average gradient between B and A is positive so the curve is increasing. Also see this by looking at the graph.

b) The average gradient between C and D is negative so the curve is decreasing. See the graph.

Exercise 6.13

1) Calculate the average gradient q between

   a) A and B

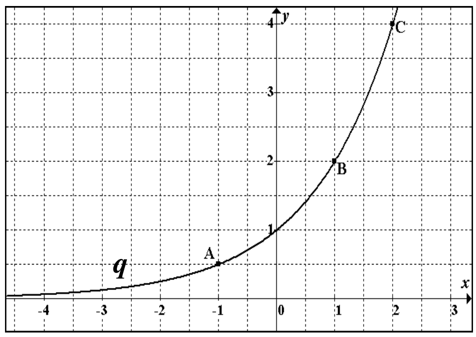
   …………………………………………

   …………………………………………

   b) B and C

   …………………………………………

   …………………………………………



2) Calculate the average gradient of f between:

   a) $x = 1$ and $x = 2$

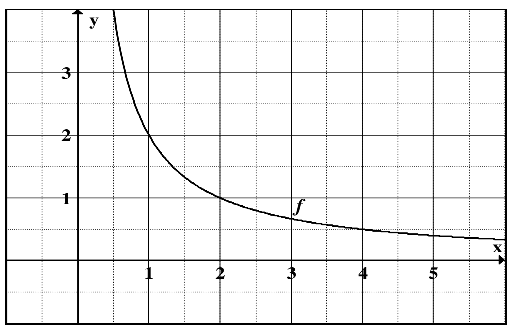
   …………………………………………

   …………………………………………

   b) $x = 2$ and $x = 4$.

   …………………………………………

   …………………………………………



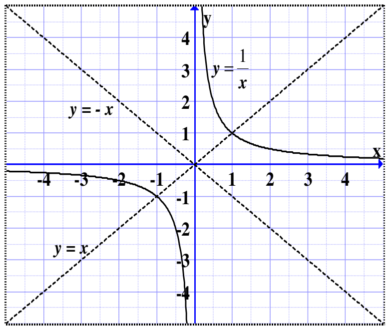
§ 6.14 DRAWING THE GRAPH OF THE HYPERBOLA

- The general form of a hyperbola is $\mathbf{y = \frac{a}{x} + q}$, $a \neq 0$
- The simplest form of the equation of the hyperbola is $y = \frac{a}{x}$.

EXAMPLE	SOLUTION
1) Draw up a table of values for x and y for the function $y = \frac{1}{x}$ and draw and label the graph	

x	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

2) On the same axes, draw and label the graphs of $y = x$ and $y = -x$



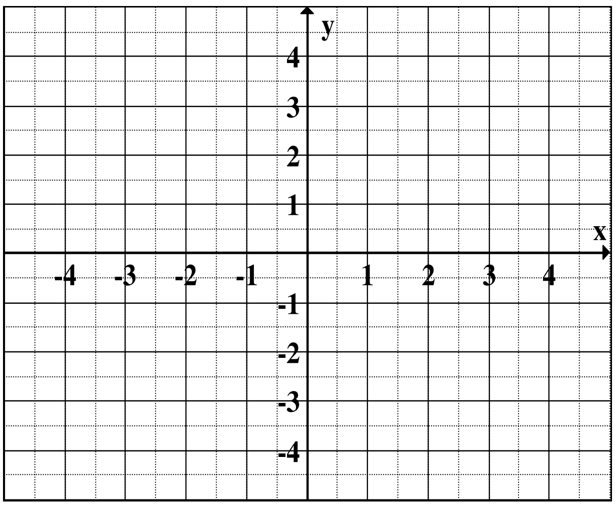
Note:

- The graph does not cut either of the axes, but becomes very close to them. The x-axis ($y = 0$) and the y-axis ($x = 0$) are called **asymptotes** to the graphs
- Each hyperbola consists of two 'curves'. When $a > 0$, these two curves lie in quadrants 1 and 3
- The graph is symmetrical about lines $y = x$ and $y = -x$

<u>Exercise 6.14</u>

On the same axes, draw and label the graphs of: 1) $y = x$ 2) $y = -x$ 3) $y = \frac{2}{x}$

x	-4	-2	-1	1	2	4
$y = \frac{2}{x}$						



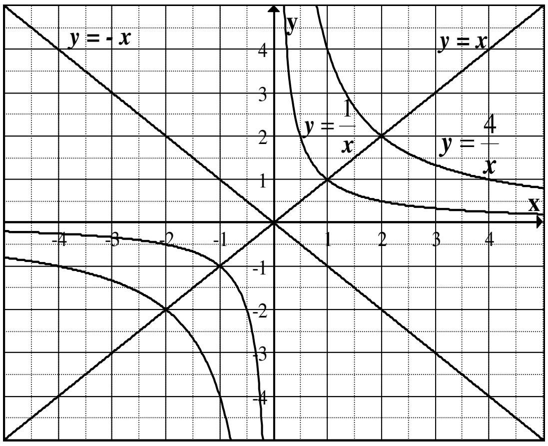
§ 6.15 THE EFFECT OF 'a' ON THE GRAPH OF THE HYPERBOLA

EXAMPLE

On the same axes, draw and label the graphs of: 1) $y = x$ 2) $y = -x$ 3) $y = \frac{1}{x}$ 4) $y = \frac{4}{x}$

SOLUTION

$y = \frac{1}{x}$		$y = \frac{4}{x}$	
x	***y***	***x***	***y***
-2	$-\frac{1}{2}$	-4	-1
-1	-1	-2	-2
$-\frac{1}{2}$	-2	-1	-4
$\frac{1}{2}$	2	1	4
1	1	2	2
2	$\frac{1}{2}$	4	1

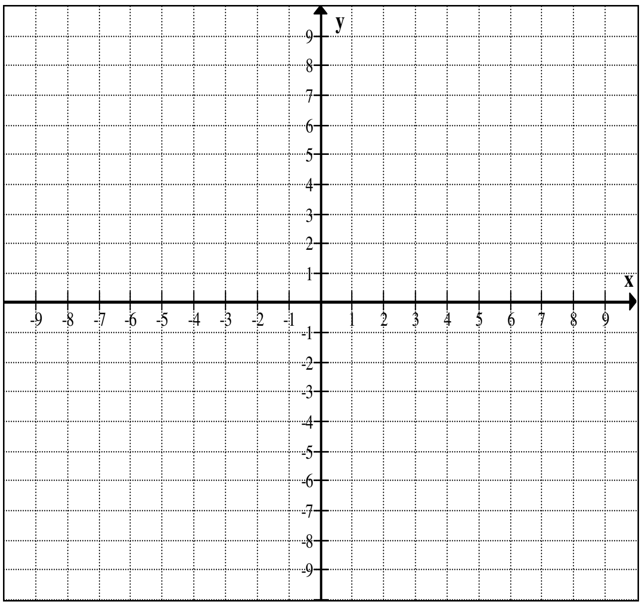


Note:
- Both graphs are symmetrical about both the lines $y = x$ and $y = -x$.
- The line $y = \frac{1}{x}$ cuts the graph of $y = x$ at $(-1; -1)$ and $(1; 1)$. The graph of $y = \frac{4}{x}$ .cuts the line $y = x$ at the points $(2; 2)$ and $(-2; -2)$.
- Since $a > 0$, the "curves" of the hyperbola lie in quadrant 1 and quadrant 3.
- As the value of a increases, the 'curves' of the graph move further from the origin
- The x-axis ($y = 0$) and the y-axis ($x = 0$) are **asymptotes** to the graphs

Exercise 6.15

On the same axes, draw and label the graphs of: 1) $y = \frac{9}{x}$ 2) $y = \frac{16}{x}$ 3) $y = x$ 4) $y = -x$

$y = \frac{9}{x}$		$y = \frac{16}{x}$	
x	***y***	***x***	***y***
-9			
-3			
-2			
-1			
1			
2			
3			
9			



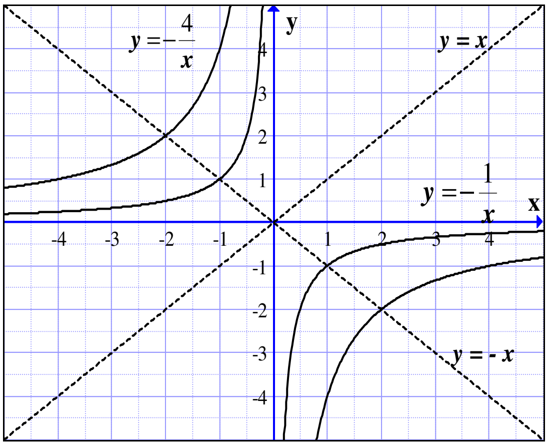
§ 6.16 DRAWING A HYPERBOLA WHEN $a < 0$

EXAMPLE

On the same axes, draw and label the graphs of: 1) $y = x$ 2) $y = -x$ 3) $y = -\frac{1}{x}$ 4) $y = -\frac{4}{x}$

SOLUTION

$y = -\frac{1}{x}$		$y = -\frac{4}{x}$	
x	***y***	***x***	***y***
-2	$\frac{1}{2}$	-4	1
-1	1	-2	2
$-\frac{1}{2}$	2	-1	4
$\frac{1}{2}$	-2	1	-4
1	-1	2	-2
2	$-\frac{1}{2}$	4	-1



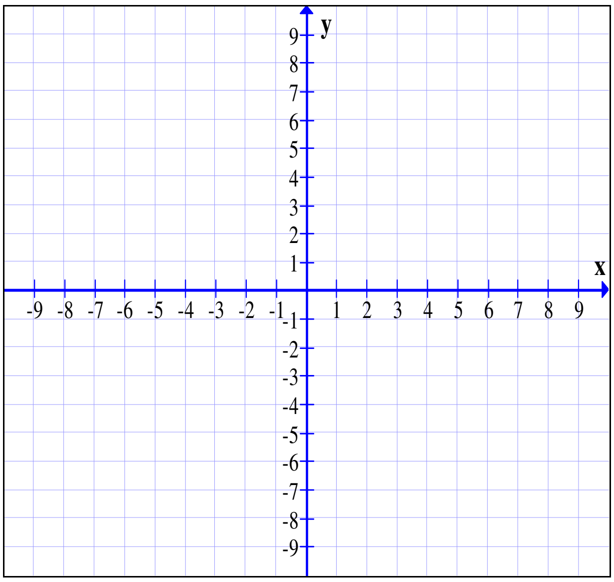
Note:

- Both hyperbolas are symmetrical about the lines $y = x$ and $y = -x$
- Since $a < 0$, the 'curves' of the hyperbola lie in quadrant 2 and quadrant 4.
- As the value of a decreases, the 'curves' of the graph move further from the origin
- The line $y = -\frac{1}{x}$ cuts the graph of $y = -x$ at $(-1; 1)$ and $(1; -1)$. The graph of $y = \frac{4}{x}$ .cuts the line $y = -x$ at the points $(-2; 2)$ and $(2; -2)$.
- The x-axis ($y = 0$) and the y-axis ($x = 0$) are **asymptotes** to the graphs

Exercise 6.16

On the same axes, draw and label the graphs of: **1)** $y = x$ **2)** $y = -x$ **3)** $y = -\frac{9}{x}$ **4)** $y = -\frac{16}{x}$

$y = -\frac{9}{x}$		$y = -\frac{16}{x}$	
x	***y***	***x***	***y***
-9			
-3			
-2			
-1			
1			
2			
3			
9			



§ 6.17 DRAWING A HYPERBOLA WHEN $q \neq 0$

✧ The general form of the equation of a parabola is y = $\frac{a}{x} + q$, $a \neq 0$.

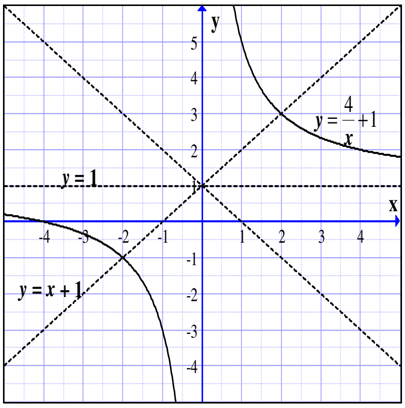
EXAMPLE

On the same set of axes draw and label the graphs of 1) $y = x + 1$ 2) $y = -x + 1$ 3) $y = \frac{4}{x} + 1$

SOLUTION

$y = \frac{4}{x} + 1$

x	y
– 4	0
– 2	– 1
– 1	–3
1	5
2	3
4	2



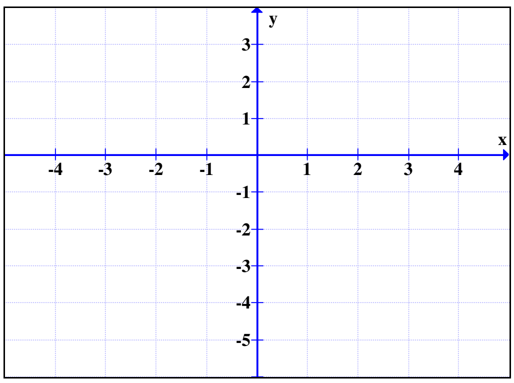
Note:

- For the hyperbola $y = \frac{4}{x} + 1$, $a = 4$ and $q = 1$.
- The y-axis ($x = 0$) is a vertical asymptote to the graph, but the x-axis ($y = 0$) is **not.** The line $y = 1$ is the horizontal asymptote.
- The graph is **not** symmetrical about the line $y = x$ and $y = -x$. The axes of symmetry are the lines $y = -x + 1$ and $y = x + 1$.
- The graph of $y = \frac{4}{x} + 1$ cuts the line $y = x + 1$ at the points (– 2; – 1) and (2; 3)
- $q = 1$ has raised the graph, one asymptote and one axis of symmetry by 1 unit

Exercise 6.17

1) On the same axes, draw and label the graphs of: a) $y = -x - 1$ b) $y = x - 1$ c) $y = \frac{4}{x} - 1$

x	– 4	– 2	– 1	1	2	4
$y = -x - 1$						
$y = x - 1$						
$y = \frac{4}{x} - 1$						



2) What is the effect of the '– 1´on the graph of $y = \frac{4}{x} - 1$? ....

3) What are the asymptotes of $y = \frac{4}{x} - 1$? ....

4) What are the axes of symmetry of $y = \frac{4}{x} - 1$? ....

§ 6.18 INTERCEPT ON THE X-AXIS OF A HYPERBOLA

- The x-intercept is the x-coordinate of the point where the graph cuts the x-axis.
- Any point on the x-axis has a y-coordinate of 0.

EXAMPLE

Find the x-intercept of each of the graphs of: 1) $y = \frac{5}{x}$ 2) $y = \frac{8}{x} + 2$ 3) $f(x) = -\frac{6}{x} - 3$

SOLUTION

1) When $y = 0$, $0 = \frac{5}{x}$

There is no value of x which divides into 5 to give zero. There is no x-intercept.

2) When $y = 0$, $0 = \frac{8}{x} + 2$

$$-2 = \frac{8}{x}$$
$$-2x = 8$$
$$\frac{-2x}{-2} = \frac{8}{-2}$$
$$x = -4$$

The x-intercept is –4.

c) When $y = 0$, $0 = -\frac{6}{x} - 3$

$$3 = -\frac{6}{x}$$
$$3x = -6$$
$$\frac{3x}{3} = \frac{-6}{3}$$
$$x = -2$$

The x-intercept is –2.

Exercise 6.18

Find the x-intercept of each of the following:

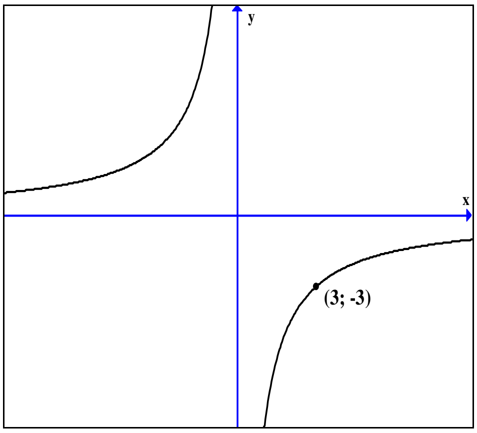
1) $y = \frac{9}{x}$ 2) $y = \frac{4}{x} + 2$ 3) $y = \frac{6}{x} - 1$

§ 6.19 FINDING THE EQUATION OF A HYPERBOLA WHEN $q = 0$

EXAMPLE

Use the given graph to write down:

1) The equation of the hyperbola
2) The equations of the axes of symmetry
3) The equations of the asymptotes.



SOLUTION

1) $$y = \frac{a}{x} + q$$

$$q = 0, \quad \therefore y = \frac{a}{x}$$

The graphs passes through (3; –3)

$$\therefore -3 = \frac{a}{3}$$

$$-9 = a$$

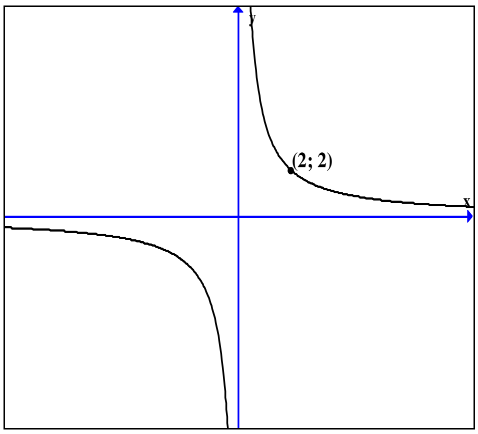
$$y = \frac{-9}{x}$$

2) The lines $y = x$ and $y = -x$ are the two axes of symmetry

3) The x-axis and the y-axis are the two asymptotes. Their equations are $y = 0$ and $x = 0$

Exercise 6.19

1) Find the equation of the given hyperbola.

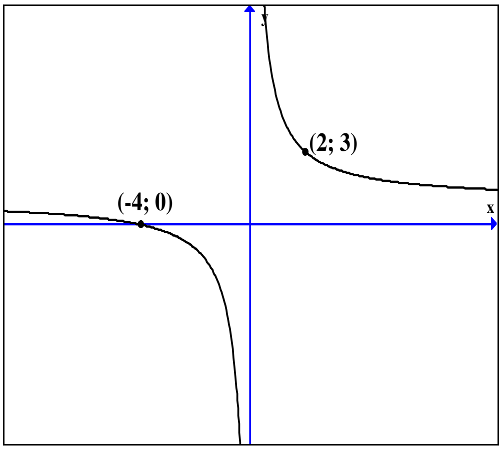


2) Give the equations of the axes of symmetry.

3) Give the equations of the asymptotes.

§ 6.20 FINDING THE EQUATION OF A HYPERBOLA WHEN $q \neq 0$

EXAMPLES



Use the above diagram to:
1) Write down the equation of the hyperbola in the form y = $\frac{a}{x} + q$, $a \neq 0$
2) Write down the equations of the axes of symmetry
3) Write down the equations of the asymptotes of the graph

SOLUTIONS

1) The given equation is $y = \frac{a}{x} + q$

Substitute (–4; 0) $\quad 0 = \frac{a}{-4} + q$

$0 = a - 4q \; ..... (i)$

Substitute (2; 3) $\quad 3 = \frac{a}{2} + q$

$6 = a + 2q \; .... (ii)$

$(ii) - (i) \quad 6 = 6q$

$\underline{q = 1}$

Substitute in (i) $\quad 0 = a - 4$

$\underline{4 = a}$

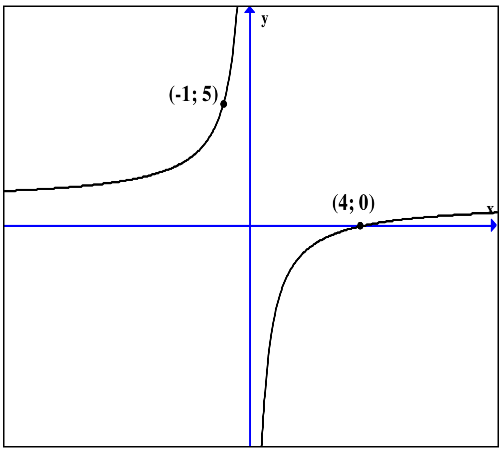
So the equation is $y = \frac{4}{x} + 1$

2) The equations of the axes of symmetry are $y = x + 1$ and $y = -x + 1$

3) The equations of the asymptotes are $x = 0$, and $y = 1$

<u>Exercise 6.20</u>

Use the adjacent diagram to



1) Write down the equation of the hyperbola which is of the form y = $\frac{a}{x} + q$, $a \neq 0$

2) Write down the equations of the axes of symmetry of the hyperbola

3) Write down the equations of the asymptotes of the hyperbola

§ 6.21 WORKING WITH NEGATIVE EXPONENTS

Remember:

- $2^3 \times 2^{-3} = 2^0 = 1$
- $2^{-3} = \frac{1}{2^3}$. Similarly $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$ and $4^{-3} = \frac{1}{4^3} = \frac{1}{64}$.
- $-3^2 = -3 \times 3 = -9$ but $(-3)^2 = -3 \times -3 = 9$
- There is a difference between
  - $2.3^{-2} = 2(3^{-2}) = 2 \times \frac{1}{3^2} = 2 \times \frac{1}{9} = \frac{2}{9}$
  - $(2.3)^{-2} = 6^{-2} = \frac{1}{6^2} = \frac{1}{36}$

EXAMPLE	SOLUTION
1) If $f(x) = 2.-3^x$, find $f(2)$	$f(2) = 2.-3^2 = 2.-9 = -18$
2) If $g(x) = -3.2^x$, calculate $g(3)$	$g(3) = -3.2^3 = -3.8 = -24$

Exercise 6.21

1) Simplify and give your answer with positive exponents:

a) 2^{-4} = ... b) 3^{-3} = ... c) 4^{-2} = ... d) 5^{-2} = ...

e) 3.2^{-2} = ... f) 4.2^{-3} = ... g) 2.3^{-2} = ... h) 3.2^{-4} = ...

i) -2.6^2 = ... j) $3.-4^{-2}$ = ... k) $2.(-3)^2$ = ... l) $-2.-3^3$ = ...

2) Complete the following tables:

a)

$f(x) = 3^x$	
x	***y***
–3	
–2	
–1	
0	
1	
2	
3	

b)

$f(x) = 2^{-x}$	
x	***y***
–3	
–2	
–1	
0	
1	
2	
3	

c)

$f(x) = 2.3^x$	
x	***y***
–3	
–2	
–1	
0	
1	
2	
3	

d)

$y = -5^x$	***x***	–3	–2	–1	0	1	2	3
	y							

e)

$y = 5^{-x}$	***x***	–3	–2	–1	0	1	2	3
	y							

§ 6.22 THE EXPONENTIAL GRAPH $y = a.b^x + q,\ a \neq 0,\ b \neq 0$

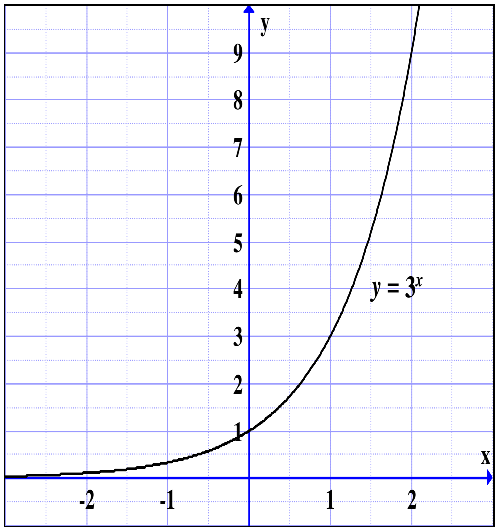
- The general form of the exponential function is $\boldsymbol{y = ab^x + q}$, $a \neq 0$ and $b \neq 0$
- The simplest form of the exponential function is $\boldsymbol{y = b^x}$, $b \neq 0$.

EXAMPLE

Complete the following table and use it to draw the graph of $y = 3^x$

SOLUTION

$y = 3^x$	
x	y
– 2	$\frac{1}{9}$
– 1	$\frac{1}{3}$
0	$3^0 = 1$
1	3
2	9

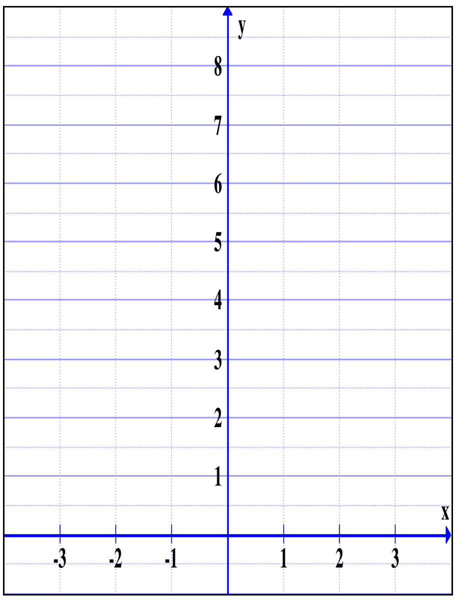


Note:
- The x-axis ($y = 0$) is an **asymptote** to the graph, so the graph does not cut the x-axis
- The graph lies in quadrants 1 and 2
- All exponential graphs of the form $\boldsymbol{y = b^x}$ will cut the y-axis at (0; 1) since any base (except zero) raised to the power of zero is 1.

Exercise 6.22

Complete the following table and use it to draw the graph of $y = 2^x$.

$y = 2^x$	
x	y
– 3	
– 2	
– 1	
0	
1	
2	
3	



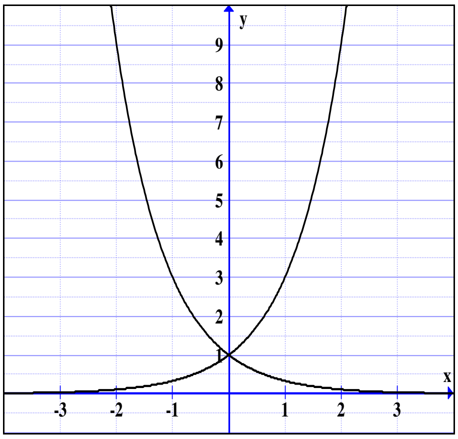
§ 6.23 DRAWING THE EXPONENTIAL GRAPH $y = b^x, b \neq 0$

EXAMPLE

Fill in the table. On the same axes, draw the graphs of $y = 3^x$ and $y = 3^{-x}$

SOLUTION

x	-3	-2	-1	0	1	2	3
$y = 3^x$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27
$y = 3^{-x}$	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$



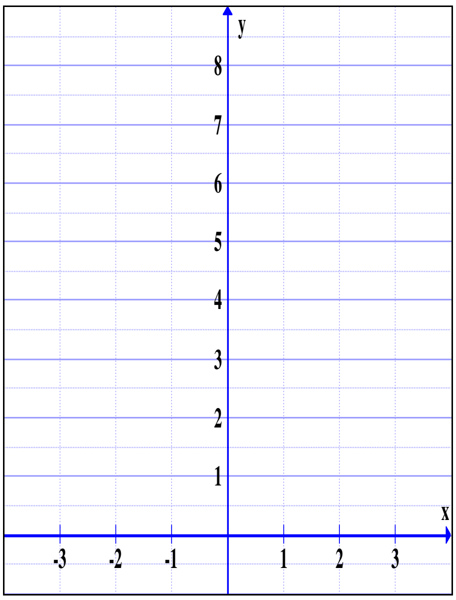
Note:

- The graphs of $y = 3^x$ and $y = 3^{-x} = \left(\frac{1}{3}\right)^x$ are symmetrical about the y-axis ($x = 0$)
- Both graphs cut the y-axis at (0; 1)
- The x-axis ($y = 0$) is an asymptote for both graphs

Exercise 6.23

Complete the following table and use it to draw the graph of $y = 2^x$ and $y = 2^{-x}$.

x	$y = 2^x$	$y = 2^{-x}$
-3		
-2		
-1		
0		
1		
2		
3		



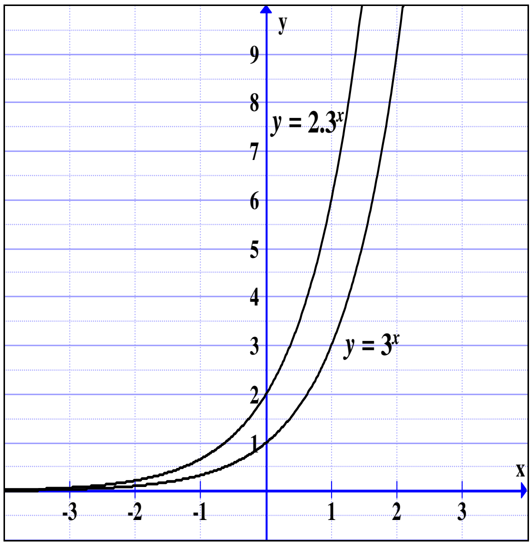
§ 6.24 DRAWING THE GRAPH OF $y = a.b^x$, $a \neq 0$ and $b \neq 0$

EXAMPLE

Complete the table and use it to draw the graphs of $y = 3^x$ and $y = 2.3^x$ on the same axes.

SOLUTION

x	$y = 3^x$	$y = 2.3^x$
– 2	$\frac{1}{9}$	$2.\frac{1}{9} = \frac{2}{9}$
– 1	$\frac{1}{3}$	$2.\frac{1}{3} = \frac{2}{3}$
0	$3^0 = 1$	$2.1 = 2$
1	3	$2.3 = 6$
2	9	$2.9 = 18$

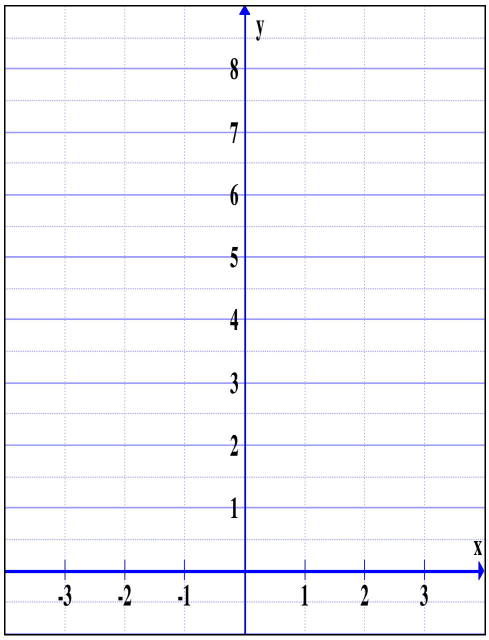


Note:
- The x-axis ($y = 0$) is an asymptote
- The y co-ordinate of $y = 3^x$ is (0; 1) whereas the y co-ordinate of $y = 2.3^x$ is (0; 2). The y co-ordinate has doubled.

Exercise 6.24

On the same axes draw and label the graphs of $y = 2^x$ and $y = 2.2^x$.

x	$y = 2^x$	$y = 2.2^x$
– 3		
– 2		
– 1		
0		
1		
2		
3		



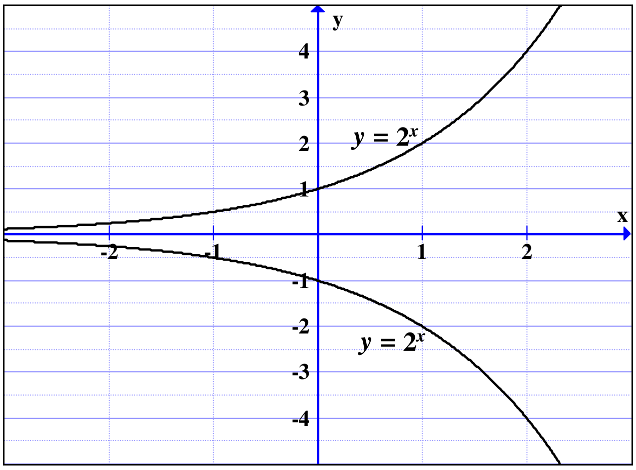
§ 6.25 DRAWING THE GRAPH OF $y = -b^x$, $b \neq 0$

EXAMPLE

On the same set of axes draw and label the graphs of $y = 2^x$ and $y = -2^x$.

SOLUTION

x	$y = 2^x$	$y = -2^x$
-2	$\frac{1}{4}$	$-\frac{1}{4}$
-1	$\frac{1}{2}$	$-\frac{1}{2}$
0	1	-1
1	2	-2
2	4	-4



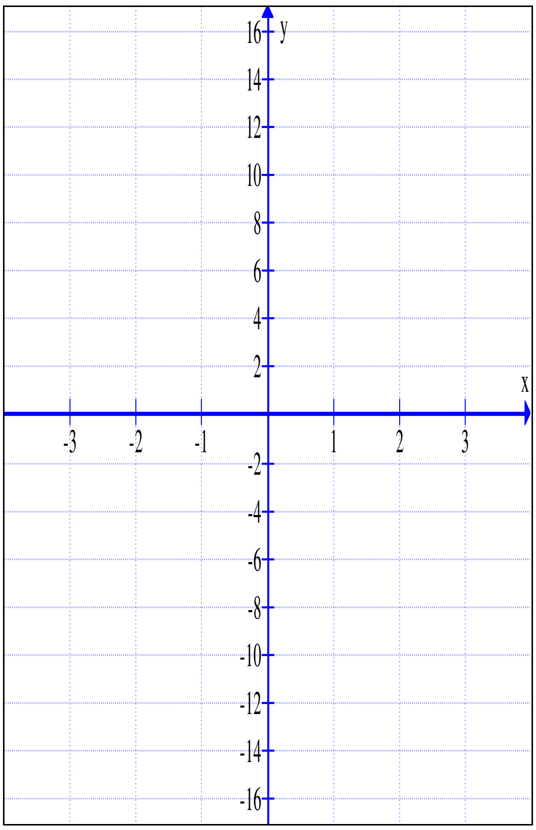
Note:

- g is a reflection of f in the x-axis
- The x-axis ($y = 0$) is the asymptote for f and g

Exercise 6.25

On the same axes draw and label the graphs of $y = 4^x$ and $y = -4^x$.

x	$y = 4^x$	$y = -4^x$
-2		
-1		
0		
1		
2		



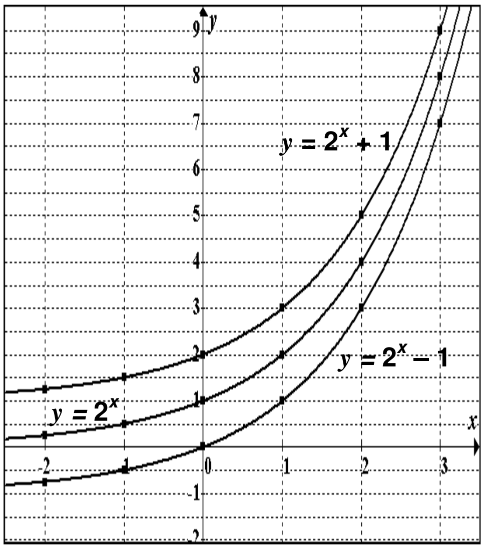
§ 6.26 DRAWING THE GRAPH OF $y = a.b^x + q,\ a \neq 0,\ b \neq 0,\ q \neq 0$

EXAMPLE

Draw up a table of values for a) $y = 2^x + 1$ b) $y = 2^x$ c) $y = 2^x - 1$.
On the same axes, draw and label these graphs.

SOLUTION

$y = 2^x + 1$		$y = 2^x$		$y = 2^x - 1$	
x	y	x	y	x	y
–2	$1\frac{1}{4}$	–2	$\frac{1}{4}$	–2	$-\frac{3}{4}$
–1	$1\frac{1}{2}$	–1	$\frac{1}{2}$	–1	$-\frac{1}{2}$
0	2	0	1	0	0
1	3	1	2	1	1
2	5	2	4	2	3
3	9	3	8	3	7



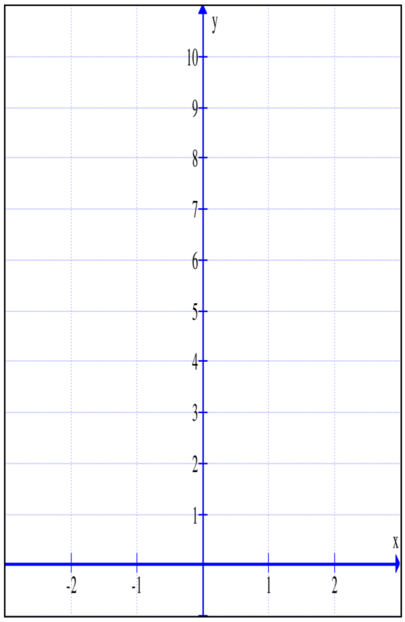
Note:

- $q = 1$ raises the graph of $y = 2^x$ by one unit. The asymptote is $x = 1$
- $q = -1$ lowers the graph of $y = 2^x$ by one unit. The asymptote is $x = -1$

Exercise 6.26

1) On the same axes, draw and label the graphs of $y = 3^x$ and $y = 3^x + 1$

x	$y = 3^x$	$y = 3^x + 1$
–2		
–1		
0		
1		
2		



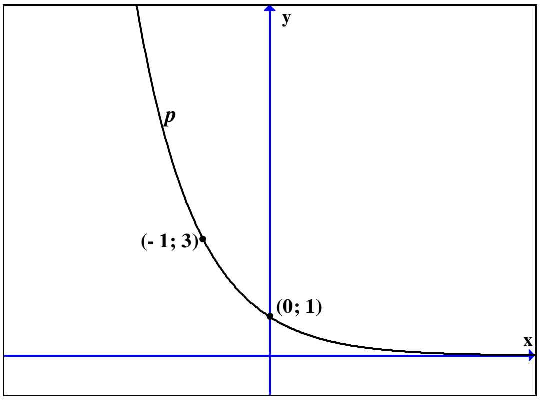
2) For each graph write down the equation of the asymptote

....

§ 6.27 FINDING THE EQUATION OF AN EXPONENTIAL FUNCTION

EXAMPLE

Find the equation of exponential graph *p*



SOLUTION

$p(x) = a.b^x + q$

(0; 1) $q = 0$

$p(x) = a.b^x$

(0; 1) $1 = a.b^0$

$1 = a$

$p(x) = b^x$

(–1; 3) $3 = b^{-1}$

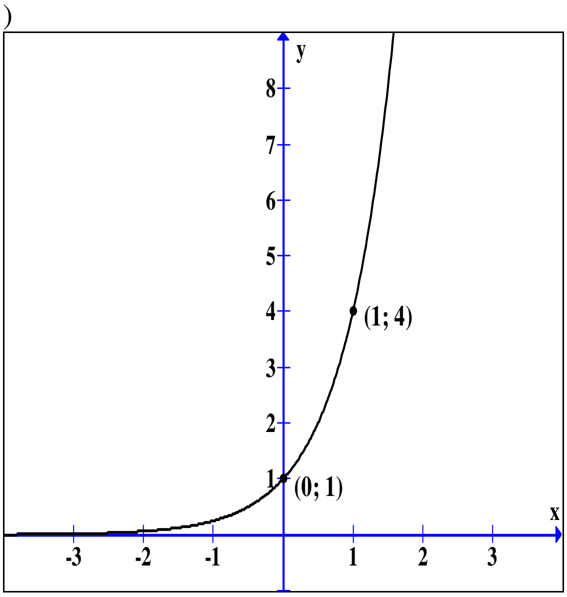
$\frac{1}{3} = b$

$p(x) = \frac{1}{3}^{x}$

Exercise 6.27

Find the equation of the following two exponential graphs

1)



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