

FINANCIAL MATHEMATICS (2)

Learning Outcomes and Assessment Standards

Learning Outcome 1: Number and number relationships

When solving problems, the learner is able to recognise, describe, represent and work confidently with numbers and their relationships to estimate, calculate and check solutions.

Assessment Standard AS 5

Demonstrate an understanding of different periods of compounding growth and decay (including effective compounding growth and decay and including effective and nominal interest rates.

Overview

In this lesson you will:

- Learn about different compounding periods
- Learn about nominal and effective interest rates.

Lesson

Different compounding periods

Compound interest is usually quoted as an annual rate. However, it can happen that interest is calculated over shorter time periods during a year.

Interest can be calculated:

- ⇒ **annually:** once per year (usually at the end of the year)
- ⇒ **half-yearly(semi-annually):** twice per year (every six months)
- ⇒ **quarterly:** four times per year (every three months)
- ⇒ **monthly:** twelve times per year (every one month)
- ⇒ **daily:** 365 times per year (excluding leap years)

Example 1

Calculate the future value of an investment of R10 000 after three years at an interest rate of 18% per annum compounded:

- (a) annually
- (b) half-yearly
- (c) quarterly
- (d) monthly
- (e) daily (excluding leap years)

- (a) Annually

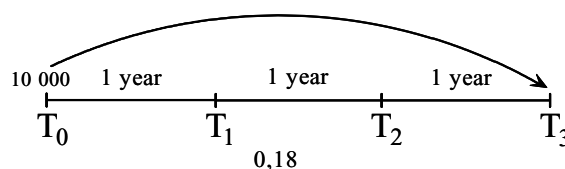
$$A = 10\,000(1 + 0,18)^3$$

$$\therefore A = 10\,000(1,18)^3$$

$$\therefore A = R16\,430,32$$

- (b) Semi-annually

$$A = 10\,000\left(1 + \frac{0,18}{2}\right)^6$$



LESSON



Overview



Lesson

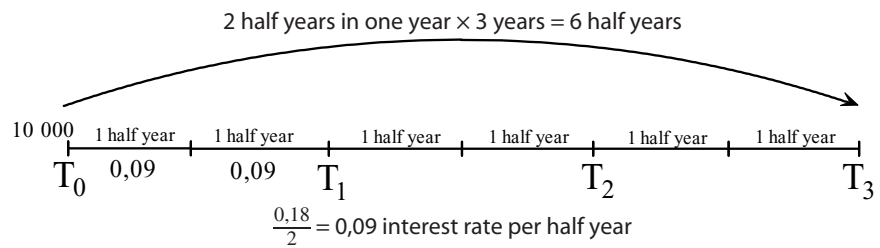


Example



$$\therefore A = 10\,000(1,09)^6$$

$$\therefore A = R16\,771$$

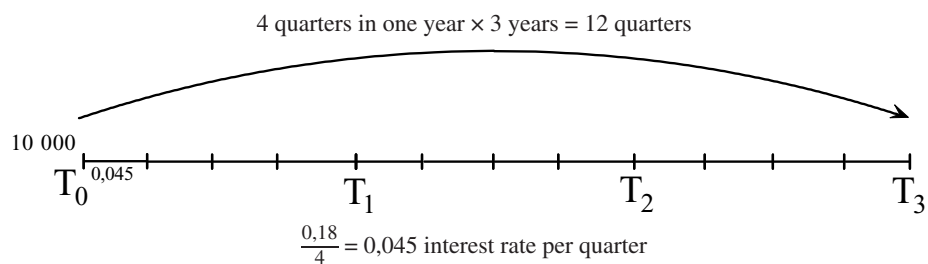


(c) Quarterly

$$A = 10\,000\left(1 + \frac{0,18}{4}\right)^{12}$$

$$\therefore A = 10\,000(1,045)^{12}$$

$$\therefore A = R16\,958,14$$



(d) Monthly

$$A = 10\,000\left(1 + \frac{0,18}{12}\right)^{36}$$

$$\therefore A = 10\,000(1,015)^{36}$$

$$\therefore A = R17\,091,40$$

(e) There are 365 days in a year (excluding leap years).

Therefore, there are $365 \times 3 = 1095$ days in three years.

The daily interest rate will be

$$\frac{0,18}{365} = 0,00049315$$

$$A = 10\,000\left(1 + \frac{0,18}{365}\right)^{1095}$$

$$\therefore A = R17\,157,78$$

Note: As the number of compounding periods increase (for the same rate) during a year, the greater the accumulated amount at the end.

Link to Activity 1

Nominal and effective interest rates

Nominal interest rates

A **nominal rate** is quoted as an annual rate, without taking into consideration the effect of compounding periods, which are shorter than the annual period. For example, 18% per annum compounded monthly is a nominal rate. The quoted annual rate is 18% but the fact that the interest is compounded monthly means that the accumulated amount at the end will be higher. The quoted annual rate of 18% (no compounding taken into account) will yield a lower accumulated amount than 18% compounded monthly.



We will refer to per annum as p.a.
 compounded quarterly c.q.
 compounded monthly c.m.
 compounded daily c.d.

Consider the following example to illustrate this.

R2 000 invested for one year at 18% per annum without monthly compounding:

$$A = 2000(1 + 0,18)^1$$

$$\therefore A = R2\ 360$$

R2 000 invested for one year at 18% per annum with monthly compounding:

$$A = 2000\left(1 + \frac{0,18}{12}\right)^{12}$$

$$\therefore A = R2391,24$$

Nominal rates:

$$r = 5\% \text{ p.a. c.m/differ}$$

OR

$$r = 18\% \text{ p.a. c.q./differ}$$

Clearly, the monthly compounding yielded a higher accumulated amount than the quoted annual rate of 18% without compounding.

It is possible to determine an annual rate that will yield the same accumulated amount as the nominal rate which was compounded monthly.

Method 1

$$A_{\text{nom}} = A_{\text{eff}}$$

$$\therefore P(1 + i_1)^{n_1} = P(1 + i_2)^{n_2}$$

$$\therefore (1 + i_1)^{n_1} = (1 + i_2)^{n_2}$$

$$\left(1 + \frac{0,18}{12}\right)^{12} = (1 + i)$$

$$\therefore i = 0,195618171$$

$$\therefore r = 19,56\% \text{ per annum compounded}$$

Method 2

The amount invested was R2 000. This accumulated to R2 391,24 by monthly compounding. So the interest gained is R391,24.

$$\text{Now: } \frac{391,24}{2000} \times 100 = 19,562$$

$$\therefore r = 19,56\% \text{ p.a. c.a.}$$

This means that a nominal rate of 18% per annum compounded monthly (p.a.c.m.) is the same as the effective rate per annum of 19,56% p.a.c.a.

Effective interest rates

As discussed previously, effective annual interest rates are equivalent annual rates that yield the same accumulated amount as rates with different compounding periods (monthly, quarterly, half yearly, daily). Effective rates are higher than quoted nominal rates. We get two types of effective rates. They are effective per period or effective annual.

For effective annual rates: (i_{eff})

$$2000(1 + 0,19562)^1 = 2000\left(1 + \frac{0,18}{12}\right)^{12} \quad (\text{both equal R2 391,24})$$

$$\therefore 2000(1 + i_{\text{eff}}) = 2000\left(1 + \frac{0,18}{12}\right)^{12}$$

$$\therefore 1 + i_{\text{eff}} = \left(1 + \frac{0,18}{12}\right)^{12}$$

We can therefore create a formula that helps to calculate the effective rate when given a nominal rate or vice versa. The formula can be written as follows:

$$1 + i_{\text{eff}} = \left(1 + \frac{i_{\text{nom}}}{n}\right)^n$$

where:

i_{eff} = effective rate (annual)

i_{nom} = nominal rate per period

n = number of compoundings per year

We have not yet established how to compare interest rates offered by two financial institutions. For example, if Bank A offers you a rate of 14% per annum compounded monthly and Bank B offers a rate of 15% per annum compounded semi-annually, which offer should we accept? The only way to really make such a choice is to “translate” both these rates into rates that read the same, that is per annum compounded annually, or per month compounded monthly. That is rate where the **stated period and the compounding periods are the same**. These rates we refer to as **effective rates**. We cannot compare them otherwise.

If an institution charges you 12% per annum, compounded monthly, we say that this is a **nominal monthly rate**, since the per annum (stated period) and the compounded monthly (compounded period) **differ**. Now if we pay 12% per annum, and the compounding took place monthly, then we will compound interest 12 times a year, since a year has twelve months. As simple as that. Thus we will pay $r = \frac{12\%}{12} = 1\%$ per month compounded monthly. This rate is referred to as an **effective rate per period**.

Note that the per annum changed to per month. The same for 12% p.a. compounded semi-annually translates into $\frac{12\%}{2} = 6\%$ per semi-annum compounded semi-annually and 12% p.a. compounded quarterly translates into $\frac{12\%}{4} = 3\%$ per quarter compounded quarterly. All of these are effective rates per period.

The problem arises when we look at translating the compounding period to a different period, such as compounded monthly to compounded weekly or compounded daily or quarterly etc. This will involve a little bit of effort, not just a simple division exercise. Remember we are not changing the rate, we merely translate it into what we require to know. The information will still have the same effect in our problems.

When we want to translate $x\%$ p.a. compounded monthly (the nominal monthly rate) to $y\%$ per semi-annum compounded semi-annually (the effective semi annual rate), or even to $z\%$ p.a. compounded semi-annually (the nominal semi-annual rate), we need to do so that both rate reflect the equivalence required. Let us try.

We wish to translate $x\%$ p.a. compounded monthly to a percentage per annum compounded semi-annually.

Since we do not work with percentages, $i_{12} = \frac{x}{100}$ and $i_2 = \frac{y}{100}$. If in one year we compound monthly. we then calculated interest 12 times that year, and semi-annual compounding will result into interest calculated twice that year. Since these rates are equivalent, and we invest P into each account, at the end of the day, each future value F will be the same.



Thus:

$$F_v = P_v \left(1 + \frac{i_{12}}{12}\right)^{12}$$

we must work with the effective monthly rate and the period becomes the amount of times we compounded in one year

$$= P_v \left(1 + \frac{i_2}{2}\right)^2$$

we must work with the effective semi-ann. rate and the period becomes the amount of times we compounded in one year

In general, the translation of the compound formula:

An amount at compound interest:

$$F_v = P_v(1 + i)^n$$

Annually:

$$F_v = P_v(1 + i)^n$$

Monthly:

$$F_v = P_v \left(1 + \frac{i_{12}}{12}\right)^{12n}$$

Semi-annually:

$$F_v = P_v \left(1 + \frac{i_2}{2}\right)^{2n}$$

Daily:

$$F_v = P_v \left(1 + \frac{i_{365}}{365}\right)^{365n}$$

Quarterly:

$$F_v = P_v \left(1 + \frac{i_4}{4}\right)^{4n}$$

Where:

$$i = \frac{r}{100}$$

n = years

P_v = Present value

F_v = Future value

Example 2

- (a) Convert a nominal rate of 18% per annum compounded monthly to an effective rate per annum.

$$1 + i_{\text{eff}} = \left(1 + \frac{i_{\text{nom}}}{n}\right)^n$$

$$\therefore 1 + i_{\text{eff}} = \left(1 + \frac{0,18}{12}\right)^{12}$$

$$\therefore i_{\text{eff}} = \left(1 + \frac{0,18}{12}\right)^{12} - 1$$

$$\therefore i_{\text{eff}} = (1,015)^{12} - 1$$

$$\therefore i_{\text{eff}} = 0,195618171$$

$$\therefore i_{\text{eff}} = 19,6\%$$

- (b) Convert an effective rate of 12,5% per annum, to a nominal rate per annum compounded half yearly.

$$1 + i_{\text{eff}} = \left(1 + \frac{i_{\text{nom}}}{n}\right)^n$$

$$\therefore 1 + 0,125 = \left(1 + \frac{i_{\text{nom}}}{2}\right)^2$$

$$\therefore 1,125 = \left(1 + \frac{i_{\text{nom}}}{2}\right)^2$$

$$\therefore (1,125)^{\frac{1}{2}} = 1 + \frac{i_{\text{nom}}}{2}$$

$$\therefore (1,125)^{\frac{1}{2}} - 1 = \frac{i_{\text{nom}}}{2}$$

$$\therefore 0,060660171 = \frac{i_{\text{nom}}}{2}$$

$$\therefore 0,121320343 = i_{\text{nom}}$$

$$\therefore r = 12,1\% \text{ per annum compounded half yearly}$$



Example

Example



Example 3

R13 000 is invested for 3 years at 15% per annum compounded monthly.

- (a) Calculate the future value of the investment using the nominal rate.

$$A = 13\,000 \left(1 + \frac{0,15}{12}\right)^{36}$$
$$\therefore A = 13\,000(1,0125)^{36}$$

$$\therefore A = R20\,331,27$$

- (b) Convert the nominal rate of 15% per annum compounded monthly to the equivalent effective rate (annual).

$$1 + i_{\text{eff}} = \left(1 + \frac{0,15}{12}\right)^{12}$$
$$\therefore i_{\text{eff}} = (1,0125)^{12} - 1$$
$$\therefore i_{\text{eff}} = 0,160754517$$

- (c) Now use the annual effective rate to show that the same accumulated amount will be obtained as when using the monthly rate.

$$A = 13\,000(1 + 0,160754517)^3$$

$$A = 13\,000(1,160754517)^3$$

$$A = R20\,331,27$$

Summary

In this lesson we demonstrated the relationship between nominal and effective rates. Nominal rates are quoted annual rates but don't take into consideration the effects of different compounding periods. The effective rate is the annual rate that yields the same accumulated amount as the calculation involving different compounding periods.

In the next Lesson, we will learn about fluctuating interest rates, withdrawals and additional payments.

Link to Activity 2

Activity



Activity 1

- Calculate the future value of an investment of R20 000 after 6 years, if the interest rate is 15% per annum compounded:
 - annually
 - half yearly
 - quarterly
 - monthly
 - daily (excluding leap years)
- How much more money would you have over a period of 15 years on a savings of R60 000 if the interest rate of 12% per annum was compounded monthly, as compared to an interest rate of 12% compounded annually?
- R30 000 is invested for 4 years at an interest rate of 16% per annum compounded quarterly. Thereafter, the accumulated amount is reinvested for a further 5 years at an interest rate of 15% per annum compounded semi-

- annually (half yearly). Calculate the value of the investment at the end of the 9 year period.
- Determine which of the following two savings options is a better investment over a period of one year, if the interest is calculated at:
 - 14% per annum compounded monthly
 - 16% per annum compounded quarterly
 - An amount of money was invested 6 years ago and it is now, after 6 years, worth R1 200 000. The interest rate for the savings period was 18% per compounded monthly. What was the amount that was originally invested 6 years ago?

Activity 2

Activity

- Convert the following nominal rates to equivalent effective rates:
 - 14% per annum compounded half yearly.
 - 16% per annum compounded quarterly.
 - 12% per annum compounded monthly.
 - 10% per annum compounded daily.
- Convert an effective rate of 14,5% per annum, to a nominal rate per annum compounded half yearly.
 - Convert an effective rate of 13,2% per annum, to a nominal rate per annum compounded quarterly.
 - Convert an effective rate of 10,5% per annum, to a nominal rate per annum compounded monthly.
- A man invests R24 000 at 16% per annum compounded quarterly for a period of 12 years.
 - Calculate the future value of the investment using the nominal rate.
 - Convert the nominal rate of 16% per annum compounded quarterly to the equivalent effective rate (annual).
 - Now use the annual effective rate to show that the same accumulated amount will be obtained as when using the monthly rate.
- Mpho deposited R500 000 into a fixed deposit savings account for a period of six years. The accumulated amount at the end of the six year period is R650 000. Calculate the interest rate paid in each of the following cases:
 - The annual effective rate.
 - The nominal rate per annum compounded monthly.