AREAS AND VOLUMES

Learning Outcomes and Assessment Standards

Learning Outcome 3: Space, shape and measurement
Assessment Standard AS 1
Use the formulae for surface area and volume of right pyramids, right cones spheres and combinations of these geometric objects.

Overview

In this lesson you will:

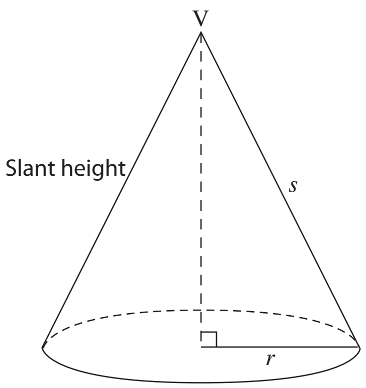
- Use the volume of a cone to solve problems
- Find the curved surface area of a cone
- Use the surface area and volume of a sphere to solve problems
- Apply this knowledge to real life problems.

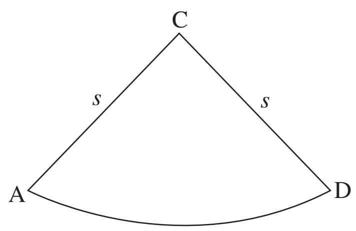
Lesson

Total surface area of a cone

If a cone of base radius *r* and slant height *s* is cut along the slant height and opened out flat, then the radius of the sector formed is *s* and the arc length AB is $2\pi r$.

The Sector ABC:





Circumference of the base = $2\pi r$ *Length of arc AB = 2πr*

Now, $\dfrac{\text{Area of sector ABC}}{\text{Area of circle with centre at C}} = \dfrac{\text{Arc length AB of sector ABC}}{\text{Circumference of circle with centre at C}}$

$\therefore \dfrac{\text{Area of sector ABC}}{\pi s^2} = \dfrac{2\pi r}{2\pi s}$

$= \dfrac{r}{s}$

Area of sector ABC $= \frac{r}{s} \times \pi s^2$

$= \pi rs$

If the curved surface area of a cone is equal to the area of sector ABC, then:

The area of the curved surface of a core $= \pi rs$

$\therefore$ Total surface area of the cone = Area of the base + Area of curved surface

$\therefore$ SA $= \pi r^2 + \pi rs$

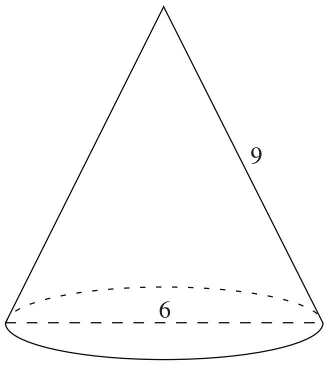
$= \pi r(r + s)$

Volume of a cone $= \frac{1}{3}$ base area $\times h$

$= \frac{1}{3}\pi r^2 h$

Examples

1. Calculate the surface area and volume of this solid



Solution

If the base diameter is 6, then the base radius (r) will be 3.

So: $s = 9$ and $r = 3$

Total surface area

$= \pi r(r + s)$

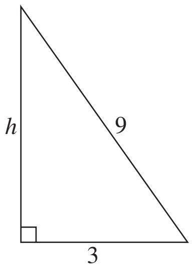
$= \pi(3)(3 + 9)$

$= 36\pi$ cm^2

$= 113{,}1$ cm^2

For the volume:– $\frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(3)^2 h$



Now AW $h^2 = 9^2 - 3^2$

$\therefore h = 8{,}5$

$\therefore$ Volume $= \frac{1}{3}\pi.9.8.5$

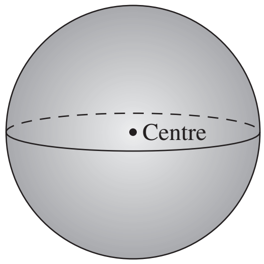
$= 79{,}97$

≈ 80 cm^3

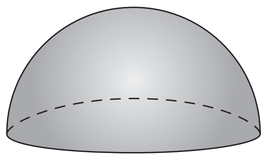
The sphere and the hemisphere

Surface area of a sphere

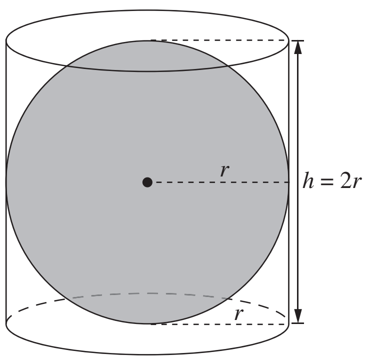
A sphere is a body bounded by a surface whose every point is **equidistant** (i.e. the same distance) from a fixed point, called the centre. For example, a shot (a heavy iron ball) is a solid sphere and a tennis ball is a hollow sphere.



One-half of a sphere is called a **hemisphere**.



Archimedes discovered that a cylinder that circumscribes a sphere, as shown in the following diagram, has a curved surface area equal to the surface area, S, of the sphere.



Surface area of sphere = curved surface area of a cylinder

$\therefore S = 2\pi rh$

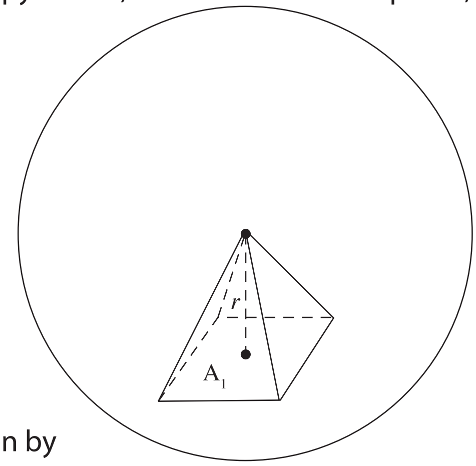
$= 2\pi r(2r)$

$S = 4\pi r^2$

Volume of a sphere

If four points on the surface of a sphere are joined to the centre of the sphere, then a pyramid of perpendicular height *r* is formed, as shown in the diagram. Consider the solid sphere to be built with a large number of such solid pyramids that have a very small base which represents a small portion of the surface area of a sphere.

If $A_1, A_2, A_3, A_4, \dots A_n$ represent the base areas (of pyramids) on the surface of a sphere, then:



V = volume of the sphere

= Sum of the volumes of all pyramids

$= \frac{1}{3}A_1r + \frac{1}{3}A_2r + \frac{1}{3}A_3r + \frac{1}{3}A_4r + \dots + \frac{1}{3}A_nr$

$= \frac{1}{3}(A_1 + A_2 + A_3 + A_4 + \dots + A_n)r$

$= \frac{1}{3}(\text{Surface area of the sphere})r$

$= \frac{1}{3} \times 4\pi r^2 \times r$

$= \frac{4}{3}\pi r^3$

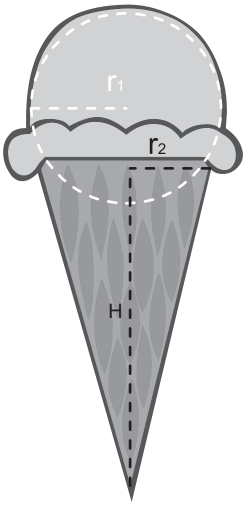
$\therefore$ The volume, *V*, of a sphere in cubic units is given by

$V = \frac{4}{3}\pi r^3$

where r is the radius of the sphere.

Example 1

Find the surface area and the volume of the hemisphere with radius 5.



Solution

a) Surface area $= 4\pi r^2 \times 0,5$

$= 4(\pi)(5)^2 \times 0,5 = 157,1\ u^2$

b) Volume $\frac{4}{3}\pi r^3 \times 0,5$ - hemisphere

$= \frac{4}{3}(\pi)(5)^3 \times 0,5 = 231,8\ u^3$

3. The radius r_1, of the ball of ice cream put into a cone is 4 cm.

The radius of the opening of the cone r_2, is 3,5 cm.

The depth of the cone, H, is 9 cm.

If the ice cream is pushed down, will the cone be able to take it?

Solution

Volume of the ball of ice cream $= \frac{4}{3}\pi(r_1)^3$

$= \frac{4}{3}\pi \cdot 4^3$

$= 268,08 \dots cm^3$

Capacity of the cone $= \frac{1}{3}A\,H = \frac{1}{3}\pi(r_2)^2.H$

$= \frac{1}{3}\pi \times 3,5^2 \times 9$

$= 346,36 \dots cm^3$

So yes, the cone will be big enough to take all the ice cream in the ball with space to spare.

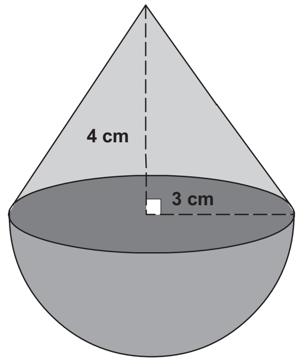
Example

Example

A child's toy is made from a cone with
height 4 cm joined to a hemisphere with radius 3 cm.

a) Calculate the volume of the toy.

b) Calculate the total surface area.



Solution

Solution

a) Volume = vol of cone + vol of hemisphere

$= \frac{1}{3}\pi r^2.H + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$

$= \frac{\pi}{3}(3^2)4 + \frac{2}{3}\pi(3)^3$

$= 12\pi + 18\pi$

$= 30\pi$

$= 94{,}3 \text{ cm}^3$

b) Total surface area = SA of hemisphere + SA of cone curved

$= \frac{1}{2}(4\pi r^2) + \pi rs$

$= 2\pi(3)^2 + \pi(3)(5)$

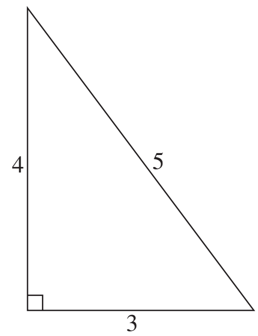
$= 18\pi + 15\pi$

$= 33\pi$

$= 103{,}7 \text{ cm}^2$

$s^2 = 16 + 9 = 25$

$\therefore s = 5$

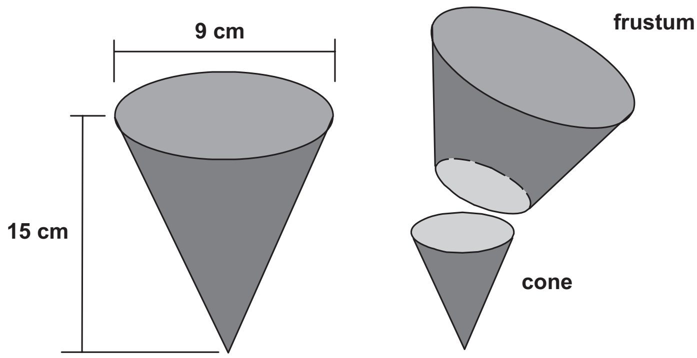


Example

Example

Eddie cuts a large wooden cone into a smaller cone and a frustum.

The smaller cone has a diameter of 6 cm and a height of 10 cm.



a) Calculate the volume of the frustum.

b) Calculate the curved surface area of the frustum.

Solution

Solution

Volume of frustum = volume of big cone - volume of small cone

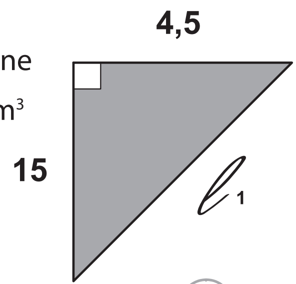
$= \left(\frac{1}{3}\right)(\pi)(4{,}5)^2(15) - \left(\frac{1}{3}\right)(\pi)(3)^2(10) = 223{,}8 \text{ cm}^3$

Curved area of big cone - curved surface area of small cone

Slant height of big cone

$s^2 = 15^2 + (4{,}5)^2$

$\therefore s_1 = 15{,}7$ u

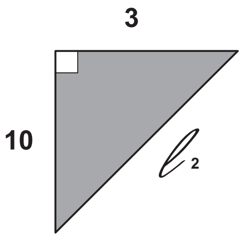


Slant height of small cone

$(S_2)^2 = 10^2 + 3^2$

$= 109$

$S_2 = 10{,}4$



Curved surface area = π (4,5)(15,7) - π (3)(10,4) = 123,9 u^2

Activity

Activity 1

1. The picture shows a model of a rural hut.

   Assume that the hut is hemispherical with a radius of 3 metres.

   Calculate:

   a) the volume of air inside the hut

   b) the outer surface area of the hut.

2. The South African Large Telescope (SALT) is situated in the Sutherland Observatory in the Western Cape. The telescope is the largest in the southern hemisphere. The housing of the telescope is in the form of a cylinder on top of which is a hemispherical geodesic dome. The radius is the cylindrical section is 13 m and its height is 17 m.

   Calculate:

   a) the outer surface area of the structure

   b) an estimate of the volume of air inside the structure.

3. The cone in the Great Zimbabwe Ruins is made of stone bricks, individually shaped from spars of granite. The measurement of each brick is roughly 35 cm by 12 cm.

   The diameter of the base of the cone is 5,5 m and its original height is estimated at 10 m.

   Calculate, assuming the dimensions given:

   a) the volume of one brick

   b) the volume of the cone

   c) an estimate of the number of bricks in the cone.

4. Rubber bungs are made by removing the tops of cones. Starting with a cone of radius 10 cm and height 16 cm, a rubber bung is made by cutting a cone of radius 5 cm and height 8 cm from the top.

   Find the volume and total surface area of the rubber bung.