LESSON 24

AREAS AND VOLUMES

Learning Outcomes and Assessment Standards

Learning Outcome 3: Shape, space and measurement
Assessment Standard
- Surface area and volume of right pyramids and cones.
- Volumes of spheres and hemispheres.

Overview

In this lesson you will:

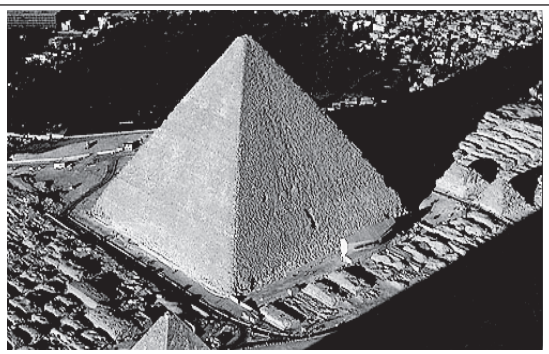
- Review what you learnt in Grade 10.
- Discover the volume of a right pyramid.
- Discover the surface area of a right pyramid.
- Apply your knowledge to problem solving.

Lesson

Activity 1

Grade 11 work

Pyramids



The Great Pyramid of Giza

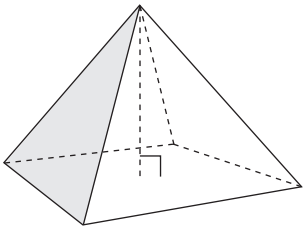
It is the one and only Wonder which does not require a description by early historians and poets. It is the one and only Wonder that does not need speculations concerning its appearance, size, and shape. It is the oldest, yet it is the only surviving of the Seven Ancient Wonders.

Defining the Pyramid

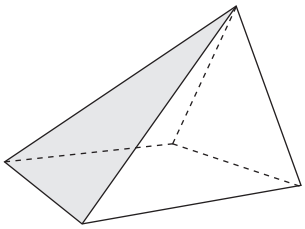
A pyramid is a polyhedron with one face (known as the "base") a polygon and all the other faces triangles meeting at a common polygon vertex (known as the "apex").

(A polyhedron is simply a three-dimensional solid which consists of a collection of polygons, usually joined at their edges)

A right pyramid is a pyramid for which the line joining the centroid of the base and the apex is perpendicular to the base.

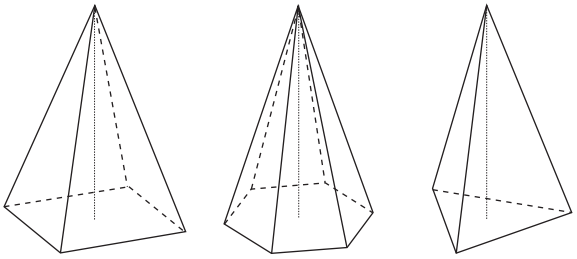


right pyramid



non-right pyramid

A regular pyramid is a right pyramid whose base is a regular polygon.



Formulae that we will use:

Volume of any regular pyramid = $\frac{1}{3} \times$ base area $\times$ height

a) The triangular based pyramid:

1. The pyramid has 4 faces
2. There are three side faces which are triangles
3. The base is a triangle
4. The pyramid has four vertices
5. The pyramid has six edges

Surface Area = [Base Area] $+ \frac{1}{2} \times$ Perimeter $\times$ [Slant Height]

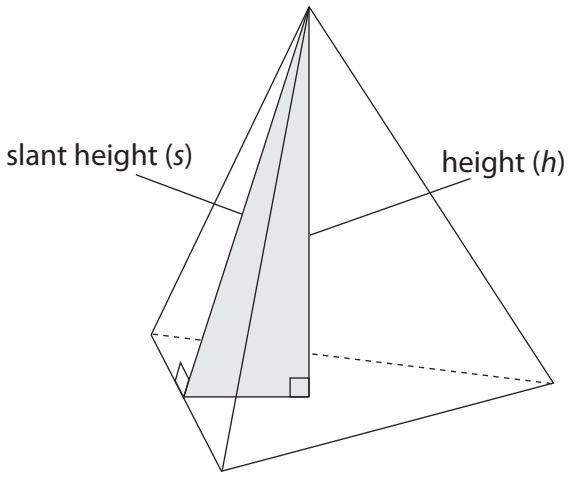
Volume $= \frac{1}{3} \times$ [Base Area] $\times$ Height

- **If all the edges are equal, we have a tetrahedron for which:**

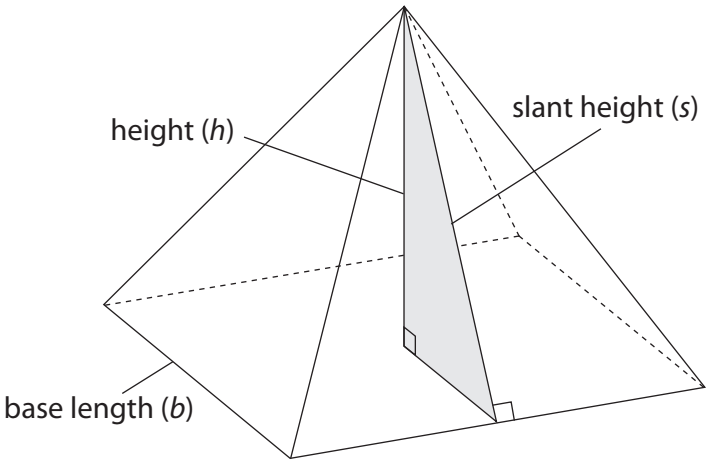
  Surface Area $= \sqrt{3} \times (\text{side length})^2$

  Volume $= \left(\frac{\text{side length}}{6\sqrt{2}}\right)^3$

- All four of the vertices in a tetrahedron are equidistant from one another



b) A square based pyramid:



1. The pyramid has 5 faces
2. There are four side faces which are triangles
3. The base is a square
4. The pyramid has five vertices
5. The pyramid has eight edges

Surface Area = [Base Area] $+\frac{1}{2}\times$ Perimeter $\times$ [Slant Height]

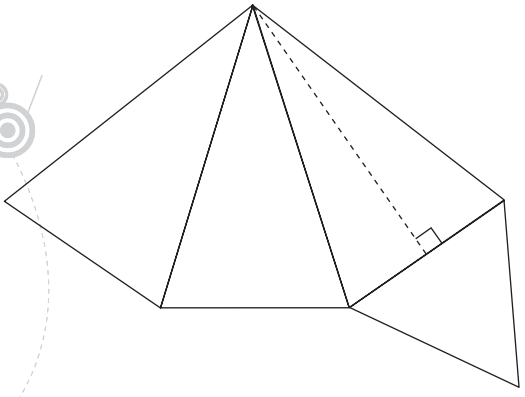
Volume $=\frac{1}{3}\times$ [Base Area] $\times$ Height

An interesting observation:

Number of base sides	Base polygon	Number of faces	Number of side faces	Number of vertices	Number of edges
3	Triangle	4	3	4	6
4	Square	5	4	5	8
5	Pentagon	6	5	6	10
6	Hexagon	7	6	7	12
7	Heptagon	8	7	8	14
8	Octagon	9	8	9	16

Where do the surface area formulas come from?

Let us look at the nets for the pyramids:



For the triangular based pyramid we have three sides each with an area of $\frac{1}{2}\times$ base $\times$ height $=\frac{1}{2}\times b\times s$.

So for the three sides we will have $3 \times \frac{1}{2} \times b \times s = \frac{3bs}{2}$. Then for the base we have one of two options:

Area rule: $\frac{1}{2} \times b \times \sin 60° = \frac{b^2}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}b^2}{4}$

Or height is $\sqrt{b^2 - \left(\frac{b}{2}\right)^2} = \sqrt{\frac{3}{4}b^2} = \frac{\sqrt{3}b}{2}$ and thus the area of the base $= \frac{1}{2} \times \frac{\sqrt{3}b}{2} \times b = \frac{\sqrt{3}b^2}{4}$.

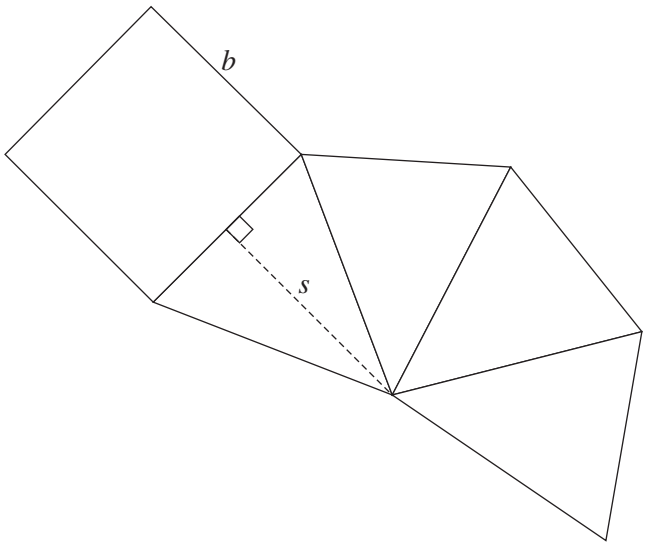
Thus the total surface area of the triangular based pyramid will be $\left(3 \times \frac{1}{2} \times b \times s\right) + \left(\frac{1}{2} \times \frac{\sqrt{3}b}{2} \times b\right) = \frac{3sb}{2} + \frac{\sqrt{3}b^2}{4}$.

That is equivalent to: Surface area of pyramid = Area of base $+ \frac{1}{2}$(perimeter of base × slant height).

Activity 1

Draw a net of the square based pyramid with base length b and slant height s and use the net to show that the formula Surface area of pyramid = Area of base $+ \frac{1}{2}$(perimeter of base × slant height) holds true in this case.

Solution Activity 1:



There are four sides which are triangular with base length b and height s. So the area of the four will collectively be $4 \times \frac{1}{2} \times$ base × height = $2bs$. The area of the base is b^2. So the total surface area of the square based pyramid will then be given by $2bs + b^2$.

Let's compare this with Area of base $+ \frac{1}{2}$(perimeter of base × slant height):

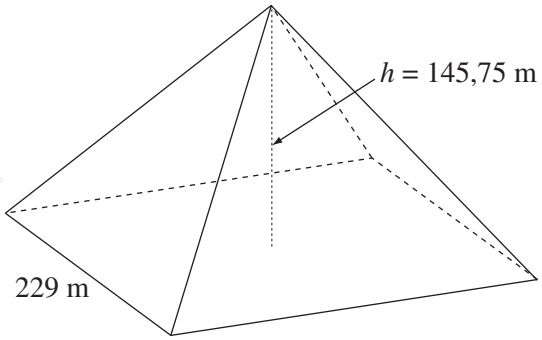
Surface area of pyramid = Area of base $+ \frac{1}{2}$(perimeter of base × slant height)

$$= b^2 + \frac{1}{2} \times 4bs \times s$$

$$= b^2 + 2bs.$$

So the formula holds true.

Example 1:

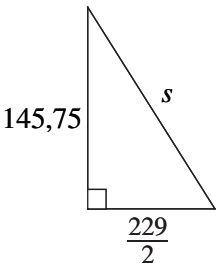


In Egypt, the Great Pyramid of Giza is 145,75 m in height and has a square base of 229 m on a side. The triangular sides angle of 51,85° with the square base.

Find the

a) Total surface area and b) volume of the great pyramid

Solution:



a) Slant height is what we need to find first:

$s = \sqrt{145{,}75^2 + \left(\frac{229}{2}\right)^2} = 185{,}35$

Or alternatively:

$s = \frac{145{,}75}{\sin 51{,}85°} = 185{,}34$

Surface area of pyramid = Area of base + $\frac{1}{2}$(perimeter of base × slant height)

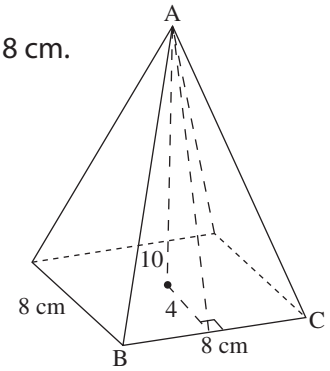
$= (229)^2 + \frac{1}{2}(4 \times 229 \times 185{,}35)$

$= 137\ 331{,}3\ m^2$

b) Volume of the pyramid = $\frac{1}{3}$ × (Base Area) × Height = $\frac{1}{3} \times 229^2 \times 145{,}75 = 2\ 547\ 758{,}58\ m^3$

Example 2

Find the surface area and volume of this pyramid with a height of 10 cm and a square base with side 8 cm.



Solution

Volume = $\frac{1}{3}$ area of base × height

$= \frac{1}{3} \times (64)(10)$

$= 213{,}3$

Surface area = area of base + 4 triangles

Height of △ABC is found by Pythagoras:

$h^2 = 4^2 + 10^2$

$\therefore h^2 = 116$

$\therefore h = 10{,}8$

Area △ABC = $\left(\frac{1}{2}\right)(8)(10{,}8) = 43{,}1$

$\therefore$ Surface Area = (8)(8) + 4(43,1)

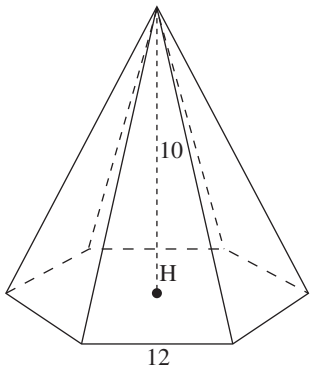
$= 236{,}3\ cm^2$

Example 3

Find the

a) volume and

b) surface area if the side of the base hexagon is 12cm, and the height (H) of the pyramid is 10 cm.



Solution

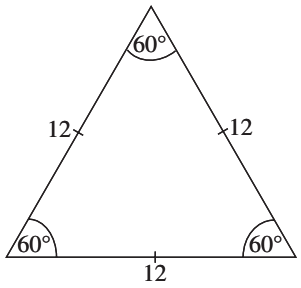
a) $V = \frac{1}{3}$ area of base $\times$ 10

The base divides into 6 congruent $\triangle$s

so area of base

$= 6 \times \left(\frac{1}{2}\right)(12)(12) \sin 60°$

$= 374{,}1 \text{ cm}^2$



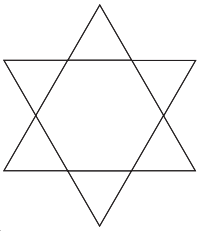
Thus:

Volume $= \frac{1}{3}.374{,}1 \times 10$

$= 1\,247 \text{ cm}^3$

b) Surface area

= Area of 6 congruent $\triangle$s + area of hexagonal base

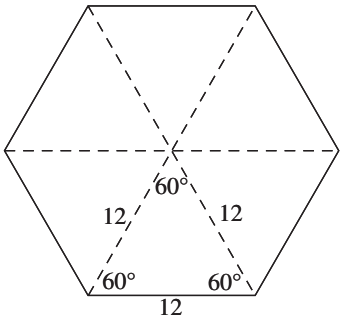


Base:

Area $= 6 \times \frac{1}{2}(12)(12) \sin 60°$

$= 3(144) \sin 60°$

$= 374{,}12 \text{ cm}^2$



$p^2 = 36 + 144 = 180$

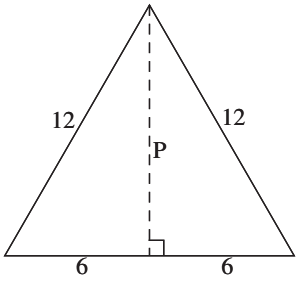
$p = 13{,}42$ cm

For the triangles:

The height of each Δ in the base.

$p^2 = 144 - 36 = 108$

$\therefore p = 10{,}39$



The slant height which will give the height of each Δ in the sides.

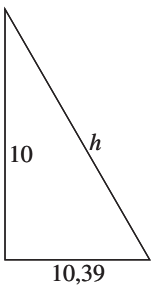
Now:

$h^2 = 10^2 + (10{,}39)^2$

$= 208$

$\therefore h = 14{,}4$

Each Δ in the sides.



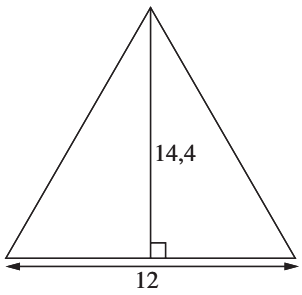
Now finally:

$6 \times \frac{1}{2}(12)(14{,}4)$

$= 518{,}4$

$\therefore$ Total surface area $= 374{,}12 + 518{,}4$

$= 892{,}52 \text{ cm}^2$



Activity 2 No 1, 2

Example 4

If it is given that the slant height of a pyramid is 22 cm and the base length is 14 cm, find the surface area and the volume of each pyramid if

a) The base is an equilateral triangle

b) The base is a pentagon

Solutions:

a) Surface area of pyramid

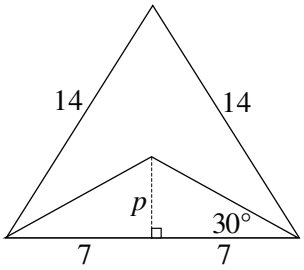
$= \text{Area of base} + \frac{1}{2}(\text{perimeter of base} \times \text{slant height})$

$= \frac{1}{2} \times 14^2 \times \sin 60° + \frac{1}{2}(3 \times 14 \times 22)$

$= 49\sqrt{3} + 462$

$= 546{,}87 \text{ cm}^2$

For the volume we need the height from the base to the apex of the pyramid.



From the top view of the base we get:

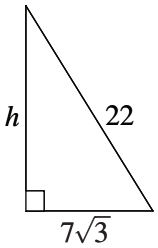
$p = \frac{7}{\tan 30} = 7\sqrt{3}$

The height is then:

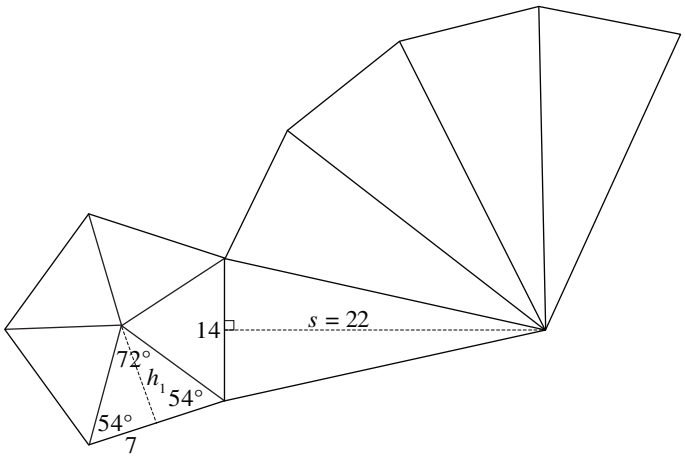
$h = \sqrt{22^2 - (7\sqrt{3})^2} = 18{,}36$

Volume of the pyramid $= \frac{1}{3} \times$ (Base Area) $\times$ Height

$= \frac{1}{3} \times \left(\frac{1}{2} \times 14^2 \times \sin 60°\right) \times 18{,}36 = 519{,}41 \text{ m}^3$



b)



For the base:

$h = 7\tan 54° = 9{,}635$ cm

Thus base area $= 5 \times \frac{1}{2} \times 14 \times 9{,}365$

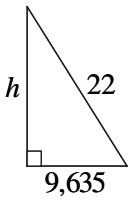
$= 337{,}23$ cm^2

Area sides $= 5 \times \frac{1}{2} \times 14 \times 22 = 770$ cm^2

Total surface area $= 770 + 337{,}23$

$= 1\ 107{,}23$ cm^2.

For the height of the pyramid:



$h = \sqrt{22^2 - 9{,}635^2} = 19{,}78$

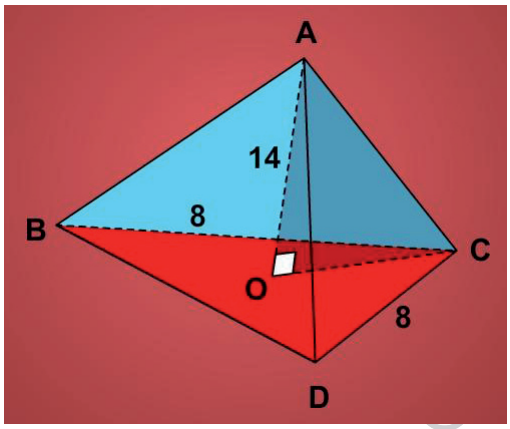
Volume of the pyramid $= \frac{1}{3} \times$ (Base Area) $\times$ Height

$= \frac{1}{3} \times 337{,}23 \times 19{,}78 = 2\ 223{,}24$ cm^3.

Example 5

Find the volume and surface area of this pyramid with a triangular base of 8 cm and a height of 14 cm.

Solution



Volume $= \frac{1}{3}$ base area $\times$ h

area of base $= \frac{1}{2}(8)(8) \sin 60°$

$= 27{,}713$ cm^2

$\therefore$ Vol $= \frac{1}{3}(27{,}713)(14)$

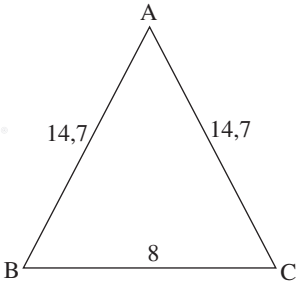
$= 129{,}3$ cm^3

Surface area = area of base + area 3 triangles

We need OC (on the ground)

Then we use Pythagoras to get AC.

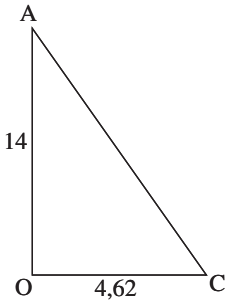
OC = 4,6



In ΔOPC:

$\cos 30° = \frac{PC}{OC}$

$\therefore OC = \frac{4}{\cos 30°} = 4{,}62$ cm



Now in ΔAOC:

$AC^2 = 14^2 + (4{,}62)^2$

$\therefore AC = 14{,}7$

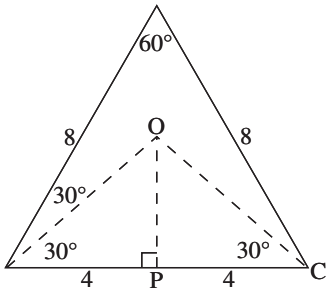
By the cos rule:

$8^2 = (14{,}2)^2 + (14{,}7)^2 - 2(14{,}7)(14{,}7)\cos A$

$2(14{,}7)(14{,}70\cos A = (14{,}7)^2 + (14{,}7)^2 - 8(2)$

$\cos A = \frac{14{,}7^2 + 14{,}7^2 - 8^2}{2(14{,}7)(14{,}7)}$

$\hat{A} = 31{,}6°$

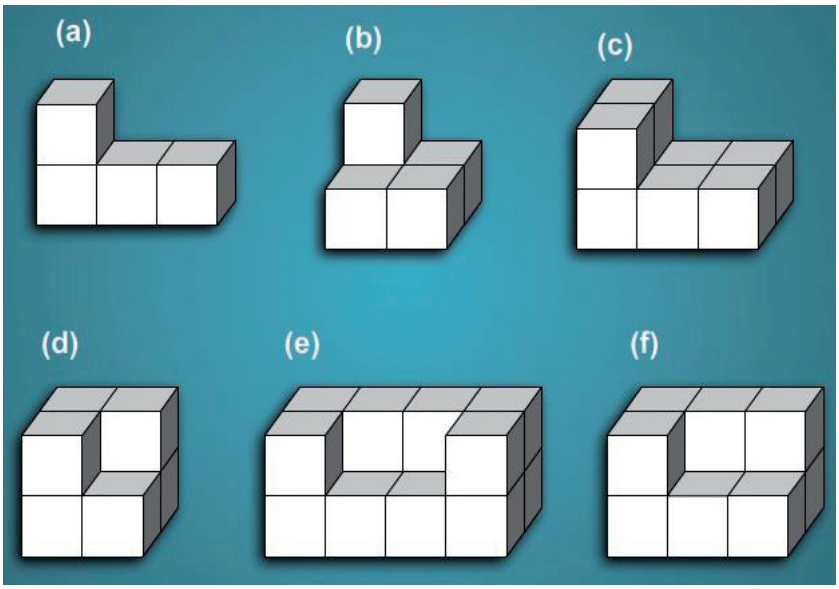


Surface area = 55,4 + 391/2)(14,7)(14,7) sin 31,6 = 225,2 cm^2

Activity 2 No 3 – 6

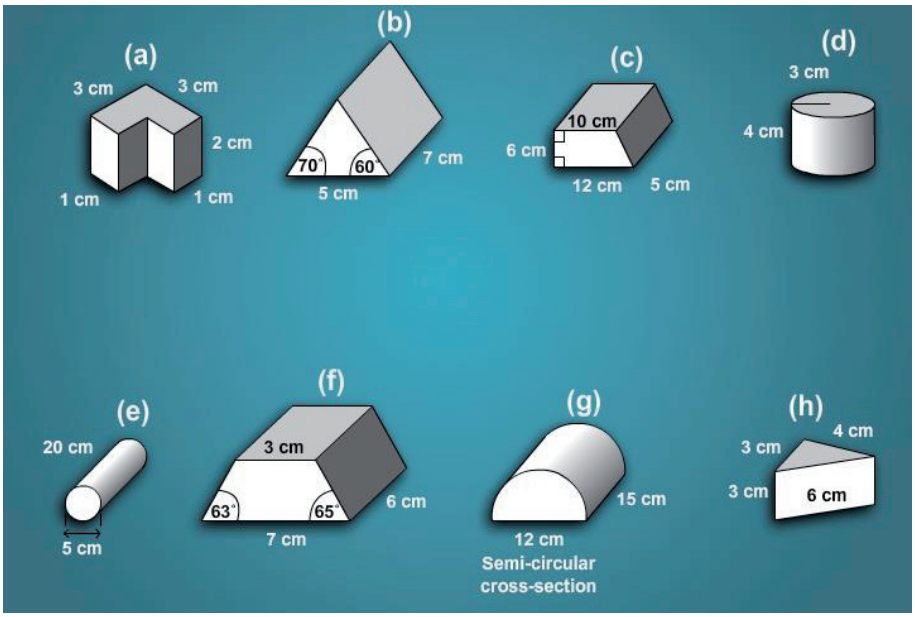
Activity 1

1. Shapes are made using one-centimetre cubes. Find the volume and surface are of each shape.



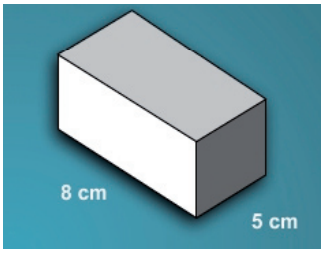
2. Find the volumes of these prisms.

Where necessary take π to be 3, 14 or use the π key on your calculator.

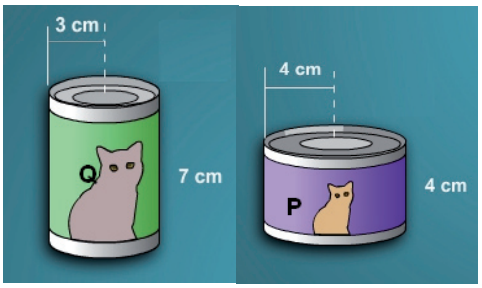


3. (a) The surface area of this cuboid is 197 cm^2

Calculate the volume of the cuboid.



(b) Which tin holds more cat food?



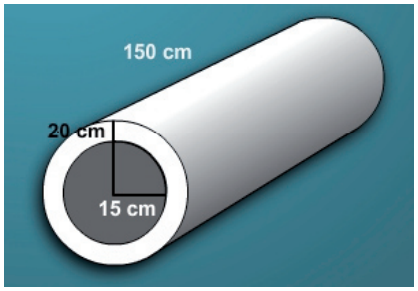
(c) A cylinder with a radius of 3 cm and a height of 8 cm is full of water. The water is poured into another cylinder with a diameter of 8 cm. Calculate the height of the water.

(d) A cylinder has a radius of 3,6 cm. the volume of the cylinder is 3346 cm^3. Calculate the total surface area of the cylinder. Give your answer to an appropriate degree of currency

(e) A concrete pipe is 150 cm long. It has an internal radius of 15 cm and an external radius of 20 cm. Calculate, giving your answer to 3 significant figures,

a) the area of the curved surface inside of the pipe.

b) the curved surface area of the outside of the pipe.



Activity 2

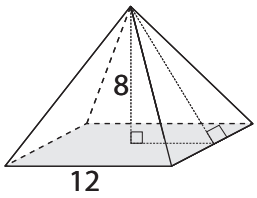
1. A pyramid of Khufu is a regular square pyramid with a base edge of approximately 776 feet and an original height of 481 feet. The limestone used to construct the pyramid weighs approximately 167 pounds per cubic foot.

   a) Estimate the weight of the pyramid of Khufu in pounds. (Assume the pyramid is a solid.)

   b) If 1 pound = 0.453 592 37 kilogram, what is the weight of the pyramid in kg?

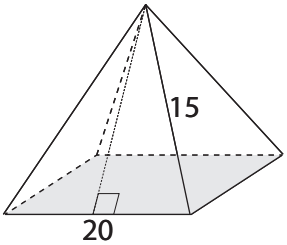
2. a. Find the length of the slant height given the base edge is 12cm and the height of the pyramid is 8cm.

   Also find the Surface area and the Volume of the pyramid.



   b. Find the length of the slant height given that the base edge is 20 cm and the lateral edge is 15 cm.

   Also find the Surface area and the Volume of the pyramid.



3. Suppose that the height of a regular square pyramid is 3 cm and the length of one edge is 5 cm. What are the surface area and volume of this pyramid?

4. Investigate what will happen to the area and volume of any square based pyramid where the base length is *b* and the height is *h*,

   i) if the base length is

      a) Halved

      b) Doubled

   ii) if the height is

      c) Halved

      d) Doubled