SOLUTIONS OF TRIANGLES

Learning Outcomes and Assessment Standards

Learning Outcome 3: Space, shape and measurement
Assessment Standard AS 5(e)
Establish and apply the sine, cosine and area rule.

Overview

Overview

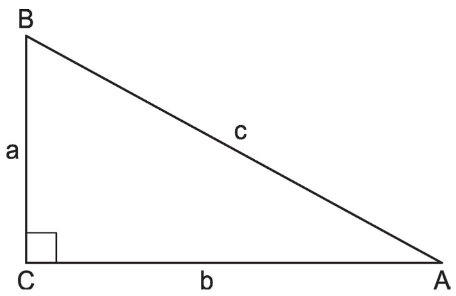
In this lesson you will:

- Review what you learnt in Grade 10
- Prove the area and sine rules
- Apply the sine and area rules to real life problems.

Lesson

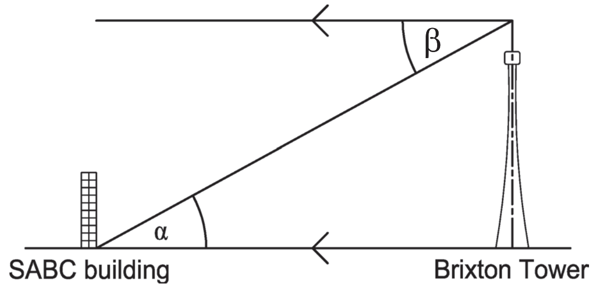
Lesson

Grade 10 work



For angle A:	For angle B:
a is the opposite side	a is the adjacent side
b is the adjacent side	b is the opposite side
c is the hypotenuse	c is the hypotenuse

Angles of elevation and depression



Angle of elevation from the bottom of the SABC building to the top of the tower is α.

The angle of depression from the top of the tower to the bottom of the SABC building is β. Why is $\alpha = \beta$?

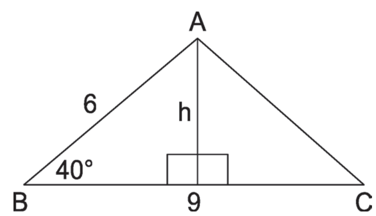
The area of a triangle is $\frac{1}{2}$ base x height. What do we do if we do not have the height?

Yes! We use trigonometry.

Example

Example 1

Find the area of $\triangle$ABC



Let's draw in height

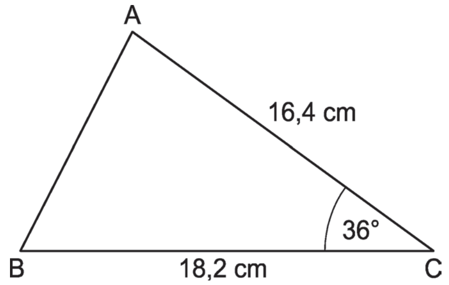
Area = $\frac{1}{2}$ base x height

base = 9 $\quad \frac{h}{6} = \sin 40°$

$\therefore$ h = 6 sin 40°

Area = $\frac{1}{2} \times 9 \times 6 \sin 40° = 17{,}36\ u^2$

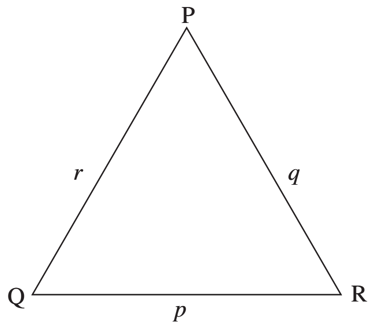
We do not need to find the height every time.



If the information is grouped so that I have two sides and the angle included by the two sides, then I can apply the area rule.

Area $= \frac{1}{2} \times 16,4 \times 18,2 \sin 36°$

$= 87,7 \text{ cm}^2$



Area $\triangle PQR \ = \frac{1}{2} pq \sin R$

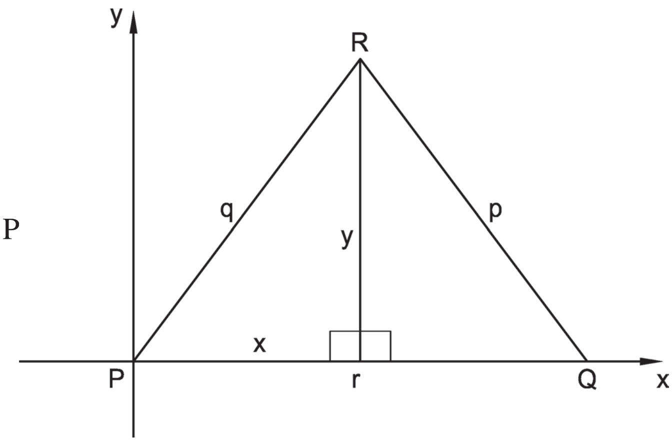
$= \frac{1}{2} rq \sin P$

$= \frac{1}{2} pr \sin Q$

Let's prove the area rule.

We use the co-ordinate system.

To find the height of the triangle, we need to find the co-ordinates of point R:



$\frac{x}{q} = \cos P$

$x = q \cos P$

$\frac{y}{q} = \sin P$

$y = q \sin P$

So R has the co-ordinates ($q \cos P$; $q \sin P$

Area $\triangle PQR \ = \frac{1}{2}$ base $\times$ height

$= \frac{1}{2} r \times y$ – co-ordinate at R

$= \frac{1}{2} \text{rq} \sin P$

Similary if R were at the origin

Area $\triangle PQR \ = \frac{1}{2} pq \sin R$

And if Q were at the origin

Area $\triangle PQR \ = \frac{1}{2} pr \sin Q$

Example

Example 2

Find the volume of this triangular prism

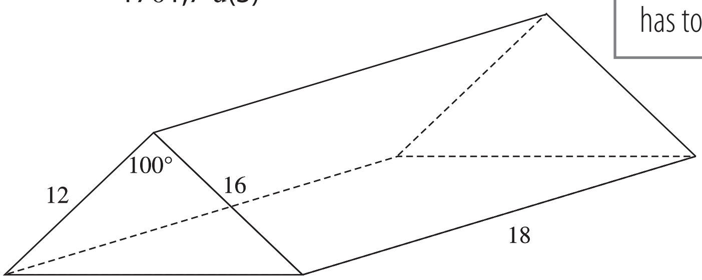
Volume = Area of base x height

= ½ x 12 x 16 sin 100° x 18

= 1701,7 u(3)

The base of the prism is a triangle and not the rectangle on which it stands.

Remember: the side we choose as our base has to be the part that is uniform.



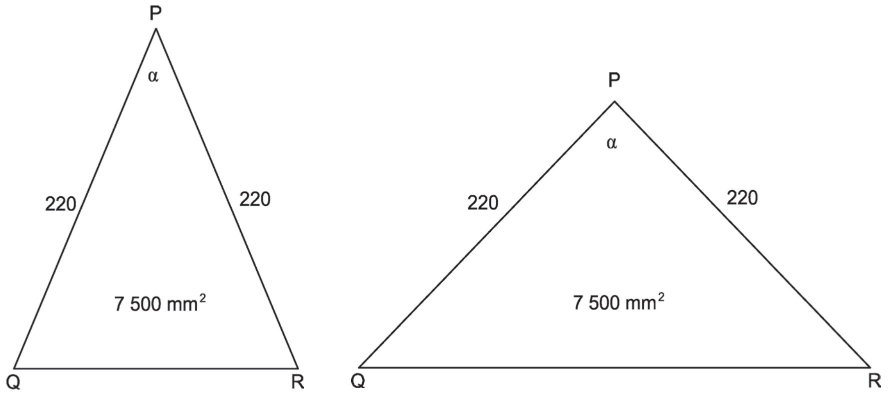
Example

Example 3

The equal sides of an isosceles triangle are 220mm. Find the possible sizes of the angles of the triangle if the area is 7500mm2.

Let's draw the triangle.

Whoops, there are two possible triangles, one in which α is acute and one which it is obtuse.



Area $= \frac{1}{2}(220)(220) \sin \alpha$

7500 $= (0{,}50(220)(220) \sin a$

$\sin \alpha = \frac{7500}{(0{,}50(220(220)}$

$\alpha = 18°$ $\alpha = 162°$

$\hat{B} = \hat{C} = 81°$ $\hat{B} = \hat{C} = 9°$

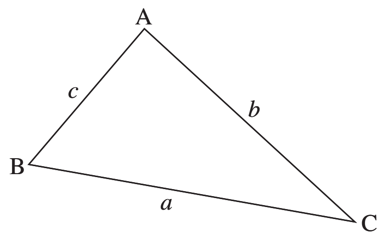
Activity

Activity 1

How do we find all the angles of a triangle that is not a right angled triangle?

We use the sine rule which tells us that

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ or $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$



Let's prove it

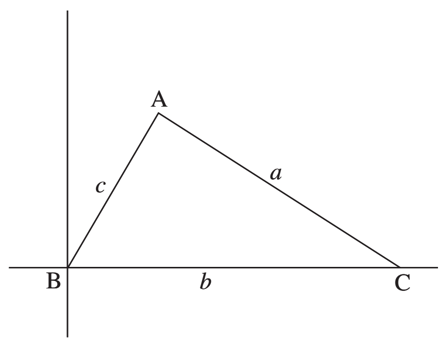
Area $\triangle ABC = \frac{1}{2}$ bc sin A

Similary if B were at the origin

Area $\triangle ABC = \frac{1}{2}$ ac sin B

And if C were at the origin

Area $\triangle ABC = \frac{1}{2}$ ab sin C



But this is the same triangle

So $\frac{1}{2}$ ab sin c $= \frac{1}{2}$ bc sin A $= \frac{1}{2}$ ac sin B

$\div \frac{1}{2}$ abc

$$\frac{\frac{1}{2}\text{ab sin C}}{\frac{1}{2}\text{abc}} = \frac{\frac{1}{2}\text{bc sin A}}{\frac{1}{2}\text{abc}} = \frac{\frac{1}{2}\text{ac sin B}}{\frac{1}{2}\text{abc}}$$

$$\frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$$

Let's use the sine rule.

Example

Example 4:

Solve $\triangle ABC$

$\hat{B} = 59°$

$$\frac{c}{\sin 100°} = \frac{12}{\sin 21°}$$

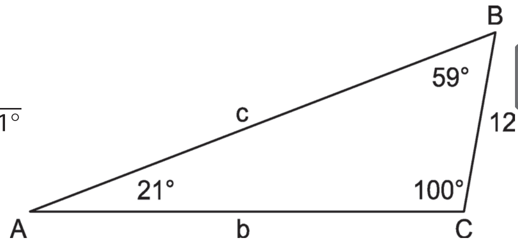
$$c = \frac{12 \sin 100°}{\sin 21°}$$

$c = 33\text{cm}$

$$\frac{b}{\sin 59°} = \frac{12}{\sin 21°}$$

$$\therefore b = \frac{12 \sin 59°}{\sin 21°}$$

$\therefore b = 28{,}7\text{cm}$



To apply the law of sines, you need to have information about two sides, and one angle opposite one of those sides, or two angles and the one side opposite one of the angles.

Activity

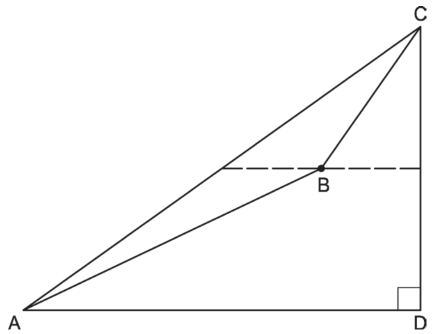
Activity 2

1. Calculate the area of a parallelogram in which two adjacent sides measures 100 mm and 120 mm and the angle between them is 65°.
2. If the area of $\triangle XYZ$ is $3000\ \text{m}^2$, $x = 80$ m and $y = 150$ m, calculate two possible sizes of Z.

Activity

Activity 3

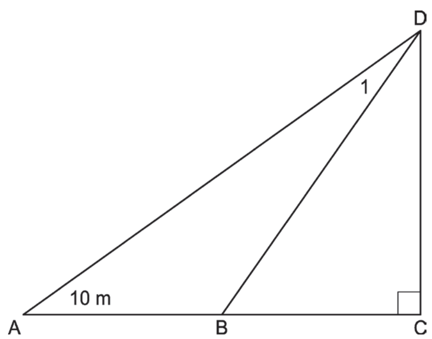
1. Look at this diagram.



At A Grant notices that the angle of elevation of the cliff top DC is 62°. He walks at an incline of 21° to point B which is 80m from A. He then notices that the angle of elevation of the top of the cliff is 78°

Find the height of the cliff.

2.



DC is a tower. The angle of elevation from A to the top of the tower is 43° and from B to the top of the tower is 74°. The distance AB is 10 m.

Find the height of the tower.

More about the sine rule

Lesson

Lesson

What happens if you have an angle and two sides, but **the side opposite the given angle** is smaller than the other given side.

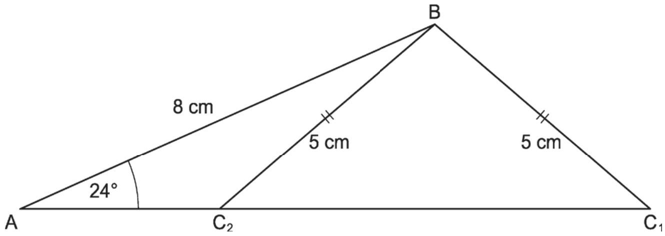
Let's investigate.

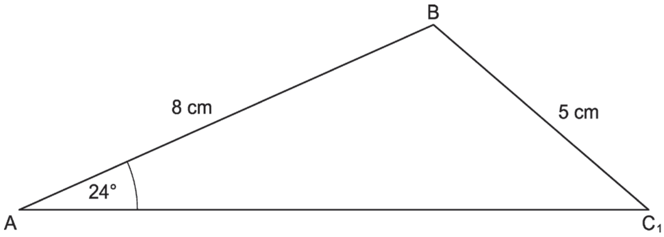
Solve $\triangle ABC$ if A = 24°

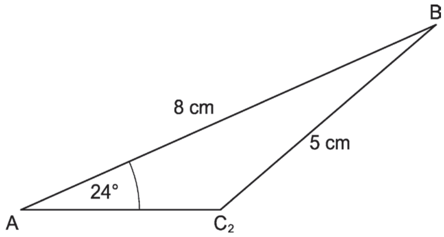
AB = 8cm and BC = 5cm

Look we have two possible triangles.

Let's draw them next to each other.







$$\frac{\sin C}{8} = \frac{\sin 24^\circ}{5}$$

$$\sin C = \frac{8\sin 24^\circ}{5}$$

$$C = 40{,}6^\circ \text{ or } C = 139{,}4^\circ$$

$$\therefore B = 115{,}4^\circ \text{ or } B = 16{,}6^\circ$$

Now:

$$\frac{b}{\sin 115{,}4^\circ} = \frac{5}{\sin 24^\circ}$$

$$b = \frac{5\sin 115{,}4^\circ}{\sin 24^\circ}$$

$$b = 11{,}1\text{cm}$$

or

$$\frac{b}{\sin 16{,}6^\circ} = \frac{5}{\sin 24^\circ}$$

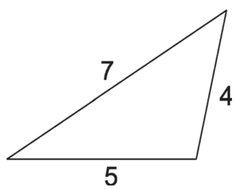
$$b = \frac{5\sin 16{,}6^\circ}{\sin 24^\circ}$$

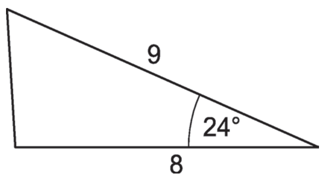
$$b = 3{,}5\text{cm}$$

This is called the ambiguous case

It happens when you are given one angle and two sides, but the shorter of the two sides is opposite the given angle.

What happens if you do not have a side opposite the given angle?





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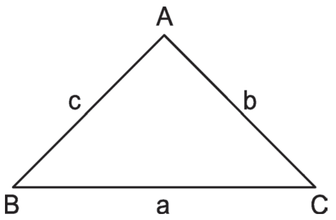
We use the cosine rule

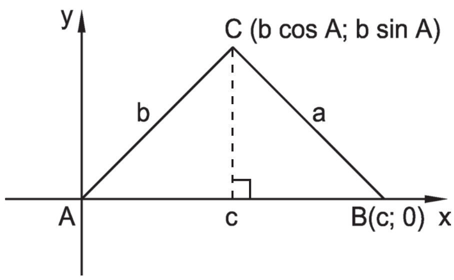


$b^2 = a^2 + c^2 - 2ac \cos B$

$a^2 = b^2 + c^2 - 2bc \cos A$

$c^2 = b^2 + a^2 - 2ab \cos C$

Let's prove it.



With A in standard position, the co-ordinates of apex C will be:

$x = b \cos A$

$y = b \sin A$

By the distance formula, we find the length of BC:-

$a^2 = (b \cos A - c)^2 + (\sin A - 0)^2$

$a^2 = b^2 \cos^2 A - 2bc \cos A + c^2 + \hat{b}^2 \sin^2 A$

$a^2 = b^2 \cos^2 A + b^2 \sin 2 A + c^2 - 2bc \cos A$

$a^2 = b^2(\cos^2 A + \sin^2 A) + c^2 - 2bc \cos A$ $\sin^2 A + \cos^2 A = 1$

$a^2 = b^2 + c^2 - 2bc \cos A$

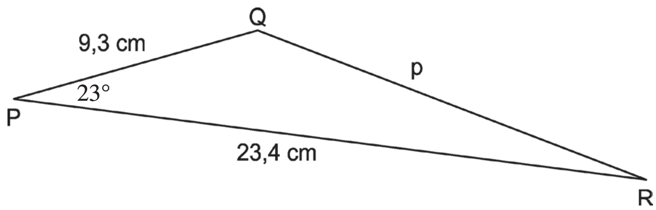
Examples

Example

1. Solve $\triangle PQR$

   $p^2 = (9,3)^2 + (23,4)^2 - 2(9,3)(23,4) \cos 23°$

   $p = 15,3$ cm

   Now use the sine rule and solve for $\hat{R}$, because we have enough information to find this angle.

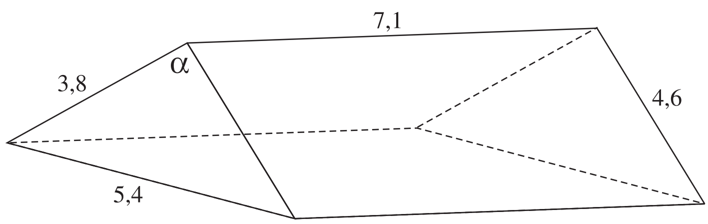
   $\frac{\sin \hat{R}}{9,3} = \frac{\sin 23°}{15,3}$

   $\sin \hat{R} = \frac{9,3 \sin 23°}{15,3}$

   $\hat{R} = 13,7$

   $\therefore \hat{Q} = 143,3°$

2. Find the volume of this triangular prism.

   Volume = are of Base × height

To use the rule we need an angle.

The cos rule

$(5{,}4)^2 = (3{,}8)^2 + (4{,}6)^2 - 2(3{,}8)(4{,}6)\cos a$

$2(3{,}8)(4{,}6)\cos\alpha = (3{,}8)^2 + (4{,}6)^2 - (5{,}4)^2$

$\cos\alpha = \dfrac{(3{,}8)^2 + (4{,}6)^2 - (5{,}4)^2}{2(3{,}8)(4{,}6)}$

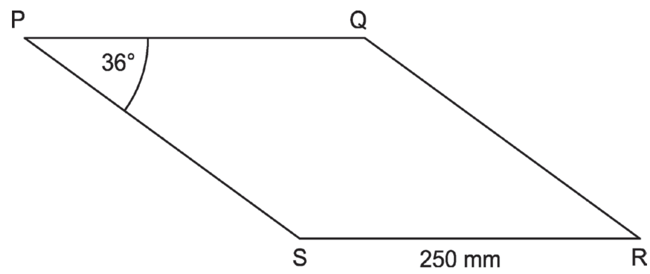
$\alpha = 79{,}4°$

Volume $= \frac{1}{2}(4{,}6)(3{,}8)\sin 79{,}4° \times \widehat{7{,}1} = 61\ \mu^3$

Activity

Activity 4

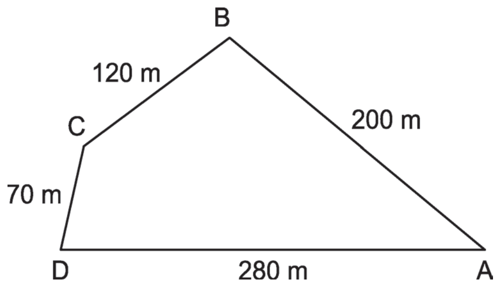
1.



PQRS is a rhombus with side 250 mm and P = 36°

Find the lengths of the diagonals

2.



Find the area of this plot of land.

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