TRIGONOMETRIC EQUATIONS

The general solution of trigonometric equations

Learning Outcomes and Assessment Standards

Learning Outcome 3: Space, shape and measurement
Assessment Standard AS 3.5(d)
Determine the general solution of trigonometric equations.

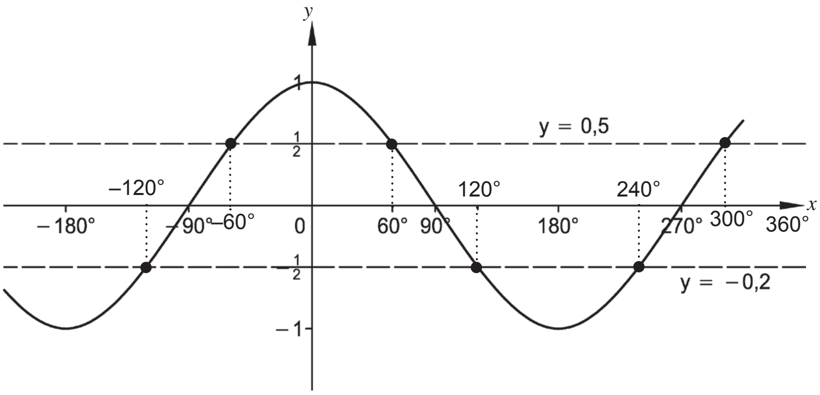
Overview

In this lesson you will:

- Establish the general solution for sin $\theta = p$, cos $\theta = p$ and tan $\theta = p$ by looking at the trigonometric graphs studied in Grade 10
- Use your calculator correctly to both establish trigonometrical ratios and angles
- Solve trigonometric equations in a given domain.

Lesson

This is the graph of $y = \cos x$ which you learnt about in Grade 10.



Let's use it to solve cos $\theta = 0{,}5$ (draw a line $y = \frac{1}{2}$ to see where the line intersects the graph).

As you can see there are many solutions. Those solutions repeat themselves every 360°, so we need to add multiples of 360°, which happens to be the period of the cos graph. So let's see what the calculator gives us.

Press [Shift cos 0,5] – you get 60° – now you have to use this to find all the other solutions.

From the graph the general solution is $\pm 60° + k \cdot 360°, k \in \mathbb{Z}$.

Let's now solve cos $\theta = –0{,}5$ (Draw a line $y = –\ 0{,}5$ to see where the line intersects the graph).

Once again there are many solutions. The calculator gives us 120°. To find all solutions we write

$\pm 120° + k \cdot 360°, k \in \mathbb{Z}$.

General solutions for cos $\theta = p$

Press [Shift cos p] to get θ and the general solution is $\pm \theta + k \cdot 360°, k \in \mathbb{Z}$

These solutions repeat themselves every 360°, so we need to add multiples of 360° (which is the period of the cosine graph)

Example

Examples

1) Solve for θ is $2\cos\theta = -0{,}646 \Rightarrow \cos\theta = 0{,}323$

Now: $\theta = \pm\cos^{-1}(-0{,}323) + k\cdot 360°$ $\quad\therefore \theta = \pm 108{,}8 + k\cdot 360°, k \in \mathbb{Z}$

$\theta = \pm 108{,}8 + k\cdot 360°, k \in \mathbb{Z}$

2) Solve for θ if $\cos 3\theta = -0{,}632$. Get the angle $\rightarrow 3\theta = \pm\cos^{-1}(-0{,}632) + k\cdot 360°$: $k \in \mathbb{Z}$

$\therefore 3\theta = \pm 129{,}2 + k\cdot 360°$.

We don't want 3θ we want θ, so $\theta = \pm 43{,}06 + k\cdot 120°$ $k \in \mathbb{Z}$. Notice here that the period of $\cos 3\theta$ is no longer 360°, but $\frac{360°}{3} = 120°$

3) Solve for θ if $\cos(\theta - 40°) = \tan 22°$.

$\cos(\theta - 40°) = [0{,}4040262258]$

Leave that answer in the calculator and press shift cos to get the angle.

$\theta - 40° = \pm 66{,}2 + k\cdot 360° \therefore \theta = 40° \pm 66{,}2° + k\cdot 360°$

$\theta = (106{,}2° \text{ or } -26{,}2°) + k\cdot 360°, k \in \mathbb{Z}$

4) Solve for θ if $\cos(\theta - 50°) = \cos 2\theta$. You have the angle. Spread out

$\cos(\theta - 50°) = \cos 2\theta$

So: $\theta - 50° = \pm 2\theta + k\cdot 360°$

$\therefore \theta = 2\theta = 50° + k\cdot 360°$

Thus $3\theta = 50° + k\cdot 360°$ or $-\theta = 50° + k\cdot 360°$

$\theta = 16{,}7° + k\cdot 120°$ or $\theta = -50° + k\cdot 360°, k \in \mathbb{Z}$

5) Solve for θ if $\cos^2 2\theta = \frac{1}{4}$

$\cos 2\theta = \frac{\pm 1}{2}$

$2\theta = \pm\cos^{-1}\left(\frac{1}{2}\right) + k\cdot 360°$ or $2\theta = \pm\cos^{-1}\left(-\frac{1}{2}\right) + k\cdot 360°$

$2\theta = \pm 60° + k\cdot 360°$ or $2\theta = \pm 120° + k\cdot 360°$

$\theta = \pm 30° + k\cdot 180°$ or $\theta = \pm 60° + k\cdot 180°, k \in \mathbb{Z}$

6) $\cos 3\theta = -\cos\theta$

We need to remove the negative by reading from the left to the right.

$\cos 3\theta = -\cos\theta$ and is thus negative.

So $\cos 3\theta < 0$, and according to the CAST rule, this happens in the 2nd and 3rd quadrants

The horizontal reduction formulae in these quadrants are $180° - \theta$ and $180° + \theta$, so we can combine them to $(180° \pm \theta)$

So $-\cos\theta = \cos(180° \pm \theta)$

Now $\cos 3\theta = \cos(180° \pm \theta)$

$\therefore 3\theta = 180° \pm \theta + k\cdot 360°$

$\therefore 3\theta \pm \theta = 180° + k\cdot 360°$

If we split:

$3\theta \pm \theta = 180° + k\cdot 360°$ $\qquad 3\theta - \theta = 180° + k\cdot 360°$

$4\theta + = 180° + k\cdot 360°$ $\qquad 2\theta = 180° + k\cdot 360°$

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$\theta + = 450° + k\cdot 90°$ … (1) $\theta = 90° + k\cdot 120°$ … (2), $k \in \mathbb{Z}$

Since $\theta \in [-180°; 90°]$

For 1: $\theta = \pm 45°; -135°$

From 2: $\theta = 90°; -30°; -150°$

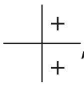
Co-ratio equations

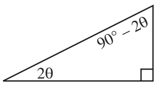
7. Solve for θ if $\cos(\theta - 10°) = \sin 2\theta$.

If this was $\cos(\theta - 10°) = \cos 2\theta$ as in a similar example (4) above, then the ratios are already balanced. So we would only focus on the angles.

Here our duty is to get those ratios the same:

Remember sine becomes cosine in vertical reduction, so $\cos(\theta - 10°) = \cos 2\theta$ and the $\sin 2\theta$ has a positive sign in front of it. So we ask where is $(\theta - 10°)$ positive?

According to the CAST rule, , it will be positive in the first and fourth quadrants.



So in the first quadrant $\sin 2\theta = \cos(90° - 2\theta)$ and in the fourth quadrant $\sin 2\theta = \cos(2\theta - 90°)$ quadrant

So: $\cos(\theta - 10°) = \cos(90° - 2\theta)$ or $\cos(\theta - 10°) = \cos(2\theta - 90°)$

$\therefore \theta - 10° = 90° - 2\theta + k\cdot 360°$ or $\theta - 10° = 2\theta - 90° + k\cdot 360°$

$3\theta = 100° + 360°$ $-\theta = -80° + k\cdot 360°$

and $\theta = \left(\frac{100}{3}\right)^{\circ} + k\cdot 120°$ $\therefore \theta = 80° + k\cdot 360°, k \in \mathbb{Z}$

Workbook: Lesson 16 Activity

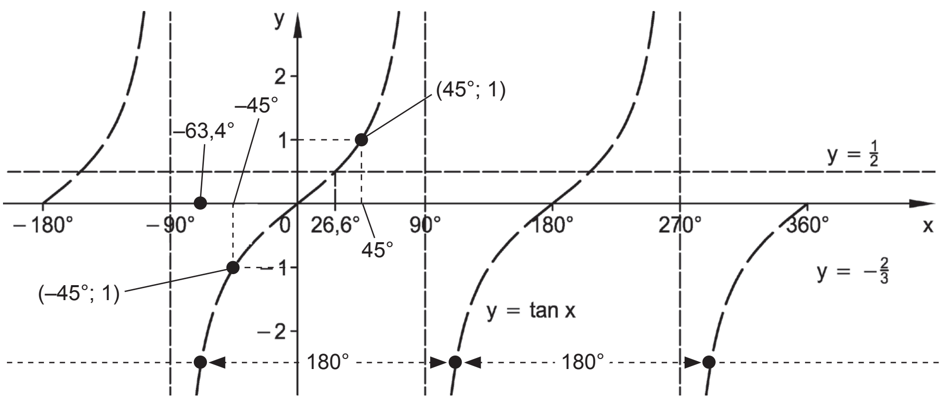
Lesson

For conclusion

This is the graph of $y = \tan x$ (you learnt to draw it in Grade 10)

Lesson 17–18

Lesson



We will use the graph to solve $\tan x = \frac{1}{2}$

Draw the line $y = \frac{1}{2}$. (Press shift tan.5, and the calculator gives us 26,6°).

We need to include all the solutions. We simply add $k\cdot 180°$ if $k \in \mathbb{Z}$ because the period of the tan graph is 180°

General solution: $x = 26{,}6° + k \cdot 180° \quad k \in \mathbb{Z}$.

Now let's solve $\tan x = -\frac{2}{3}$ (Draw the line and see what the calculator gives you, then $+ k \cdot 180° \quad k \in \mathbb{Z}$)

so $x = -63{,}4° + k \cdot 180° \quad k \in \mathbb{Z}$

General solution for $\tan \theta = p$

$\therefore \theta = \tan^{-1}(p) + k \cdot 180° k \in \mathbb{Z}$

$k \in \mathbb{Z}$

Example

Worked examples

1. Solve for θ

   $\tan 3\theta = -2{,}7$

   Press Shift tan (–2,7) to get the angle and write down the general solution.

   $3\theta = -69{,}7 + k \cdot 180°$ Divide by 3

   $\theta = -23{,}2 + k \cdot 60°, k \in \mathbb{Z}$

2. Solve for θ if $\theta \in [-180° ; 180°]$

   $\tan^2\theta = 0{,}81$

   $\tan \theta = \pm 0{,}9$ Split them

   $\tan \theta = 0{,}9$ or $\tan \theta = -0{,}9$ Find angles for the general solution

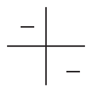
   $\theta = 42° + k \cdot 180°$ or $\theta = -42° + k \cdot 180°, k \in \mathbb{Z}$

   Substitute for k to find angles between $[-180° ; 180°]$

   $\{-138° ; -42° ; 42° ; 138°\}$

3. Solve for θ if $\tan 3\theta = -\tan(\theta + 20°)$

   $\tan 3\theta = -\tan(\theta + 20°)$

   $\tan 3\theta < 0$

   So $\tan 3\theta = \tan(180° - \theta - 20°)$

   $3\theta = 160° - \theta + k \cdot 180°$

   $\therefore 4\theta = 160° + k \cdot 180°$

   $\theta = 40° + k \cdot 45°, k \in \mathbb{Z}$

   $\tan 3\theta = \tan(-\theta - 20°)$

   $3\theta = -\theta - 20° + k \cdot 180°$

   $\therefore 4\theta = -20° + k \cdot 180°$

   $\theta = -5° + k \cdot 45°$

   NB: Can you see that both answers are in fact the same? This is important, and that happens because of the period of the tan graph. So we only need to look at one quadrant.

4. Solve for x if $3 \sin 2x = -2 \cos 2x$

   $\frac{\sin 2x}{\cos 2x} = \frac{-2}{3}$

   $\tan 2x = \frac{-2}{3}$

$2x = -33{,}7° + k \cdot 180°$

$x = -16{,}85° + k \cdot 90°$

Solution

General solution for sin θ = p

sin θ is +ve in quadrants 1 and 2

So if $\sin\theta = p$

then:

$$\theta = \begin{Bmatrix} \sin^{-1}(p) \\ 180° - \sin^{-1}(p) \end{Bmatrix} + k \cdot 360°;\ k \in \mathbb{Z}$$

Example

Example 1

Find the general solution for θ in $\sin\theta = \frac{1}{2}$:

$$\sin\theta = \frac{1}{2}$$

$$\therefore \theta = \begin{Bmatrix} \sin^{-1}\left(\frac{1}{2}\right) \\ 180° - \sin^{-1}\left(\frac{1}{2}\right) \end{Bmatrix} + k \cdot 360°;$$

$$\therefore \theta = \begin{Bmatrix} 30° \\ 150 \end{Bmatrix} + k \cdot 360°, k \in \mathbb{Z}$$

Example

Example 2

$$\sin 2\theta = -\frac{3}{4}$$

$$\therefore 2\theta = \begin{Bmatrix} \sin^{-1}\left(-\frac{3}{4}\right) \\ 180° - \sin^{-1}\left(-\frac{3}{4}\right) \end{Bmatrix} + k \cdot 360°$$

$$\therefore 2\theta = \begin{Bmatrix} -48{,}6° \\ 180°(-48{,}6°) \end{Bmatrix} + k \cdot 360°$$

$$\therefore 2\theta = \begin{Bmatrix} -48{,}6° \\ 228{,}6° \end{Bmatrix} + k \cdot 360°$$

$$\therefore \theta = \begin{Bmatrix} -24{,}3° \\ 114{,}3° \end{Bmatrix} + k \cdot 180°, k \in \mathbb{Z}$$

Example

Example 3

Solve for θ if $\sin 2\theta = -0{,}8$

$2\theta = -53{,}13° + k \cdot 360°$ or $2\theta = 180° - (-53{,}13°) + k \cdot 360°$

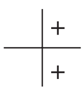
$\theta = -26{,}56° + k \cdot 180°$ or $2\theta = 233{,}13° + k \cdot 360°$

$\theta = 116{,}56° + k \cdot 180°, k \in \mathbb{Z}$

Co-ratio equations

Solve for x if $\sin 2x = \cos 3x$

In front of the cos, there is a '+' sign

So $\sin 2x > 0$ and according to the CAST rule, 

, this is in the first and second quadrants.

So we need to make the cosine a sine, and we can only do this through vertical reduction. That is $\cos 3x$ becomes $\sin(90° \pm 3x)$

So: $\sin 2x = \sin(90° \pm 3x)$

$\therefore 2x = 90° \pm 3x + k\cdot360°$

$\therefore 2x - 3x = 90° + k\cdot360°$ or $2x + 3x = 90° + k\cdot360°$

$-x = 90° + k\cdot360°$ $\quad$ $5x = 90° + k\cdot360°$

$\therefore x = -90° + k\cdot360° x = 18° + k\cdot72°; k \in \mathbb{Z}$

Trigonometric equations involving factorisation.

a) Common factor problems are those that usually have two terms.

Example

Example

Solve for x: $3 \cos x \sin x = 2 \cos x$

Do not divide by $\cos x$ because you will lose solutions

$3 \cos x \sin x - 2 \cos x = 0.$ Write in the form where one side is zero.

$\cos x (3 \sin x - 2) = 0.$ Take out a common factor

$\cos x = 0$ or $\sin x = \frac{2}{3}$

$x = \pm 90° + k\cdot360°$ or $x = 41{,}8° + k\cdot360°$ or $x = 138{,}2° + k\cdot360°$ $\quad k \in \mathbb{Z}$

b) Trinomials (3 terms)

$a^2 - a - 2$ is a quadratic trinomial that we can factorise.

So is $a^2 - ab - b^2$

In trigonometry we sometimes need to create trinomials.

Example

Example

1. Solve for θ if $2 \sin^2\theta + 5 \cos\theta + 1 = 0$

   Look at the non–squared term.

   We need a trinomial in terms of cos θ – that is with only cos and no sine terms.

   Use $\sin^2\theta = 1 - \cos^2\theta$

   $2(1 - \cos^2\theta) + 5 \cos\theta + 1 = 0$ Simplify

   $2 - 2 \cos^2\theta + 5 \cos\theta + 1 = 0$

   $-2 \cos^2\theta + 5 \cos\theta + 3 = 0$ Change signs

   $2 \cos^2\theta - 5 \cos\theta - 3 = 0$ Factorise. → (This is similar to $2x^2 - 5x - 3 = 0$ where $x = \cos\theta$.)

   $(2\cos\theta + 1)(\cos\theta - 3) = 0$

   $\therefore 2 \cos\theta = -1$ or $\cos\theta = 3$

   $\therefore 2 \cos\theta = -1$ or $\underbrace{\cos\theta = 3}_{\text{invalid}}$

   $\therefore \cos\theta = -\frac{1}{2}$

   $\therefore \theta = \pm \cos^{-1}\left(-\frac{1}{2}\right) + k\cdot360°$

   $= \pm 120° + k\cdot360°$

This is a quadratic trinomial with middle term $-3 \sin\theta \cos\theta$. The algebraic equivalent is: $a^2 + 2ab + b^2 = 0$. So we have $\cos^2\theta$, and $3 \sin\theta \cos\theta$ and we now need a $\sin^2\theta$ to complete the trinomial. The only way we can bring that in is by using the 1, since = $\sin^2\theta + \cos^2\theta$.

So $\cos^2\theta - 3\sin\theta\cos\theta + \sin^2\theta + \cos^2\theta = 0$

2. Solve for θ if $\theta \in [-180°\ ;\ 360°]$ and

$\cos^2\theta - 3\sin\theta\cos\theta + 1 = 0$

$\therefore \cos^2\theta - 3\sin\theta\cos\theta + \sin^2\theta + \cos^2\theta = 0$

$\therefore 2\cos^2\theta - 3\sin\theta\cos\theta + \sin^2\theta = 0$

$\therefore (2\cos\theta - \sin\theta)(\cos\theta - \sin\theta) = 0$ Factors

$\therefore 2\cos\theta = \sin\theta$ or $\cos\theta = \sin\theta$ Create tan θ

$2 = \frac{\sin\theta}{\cos\theta}$ or $\frac{\sin\theta}{\cos\theta} = 1$

$\tan\theta = 2$ or $\tan\theta = 1$

$\theta = 63{,}4° + k\cdot 80°$ or $\theta = 45° + k\cdot 180°, k \in \mathbb{Z}$

$\{-116{,}6°; -135°; 45°; 63{,}4°; 225°; 243{,}4°\}$

Equations with four terms: Group in two's

Example

Example

Solve for x if $2\sin^2 x + 2\sin x\cos x + \sin x + \cos x = 0$

$2\sin x(\sin x + \cos x) + (\sin x + \cos x) = 0$

$(\sin x + \cos x)(2\sin x + 1) = 0$

$\sin x = -\cos x$ or $\sin x = -\frac{1}{2}$

$\tan x = -1$

$x = -45° + k\cdot 180°$ or $x = -30° + k\cdot 360°$ or $x = 210° + k\cdot 360$, $k \in \mathbb{Z}$

Your Fact File

- If $\sin\theta = p$ and –1 £ p £ 1,
  then $\theta = \left\{\begin{matrix}\sin^{-1}(p) \\ 180° - \sin^{-1}(p)\end{matrix}\right\} + k\cdot 360°$, $k \in \mathbb{Z}$
- If $\cos\theta = p$ and $-1 \le p \le 1$,
  then $\theta = \pm\cos^{-1}(p) + k\cdot 360°$, $k \in \mathbb{Z}$
- If $\tan\theta = p$ and $p \in \mathbb{R}$,
  then $\theta = \tan^{-1}(p) + k\cdot 180°$, $k \in \mathbb{Z}$
- $1 = \sin\theta + \cos^2\theta$
- If equations have 2 terms
  - You look for a common factor
  - You may have to use co-ratios
  - You may have to form the tan ratio if each side has a cos and sine of the same angle
- If equations have three terms, it is usually a quadratic trinomial
- If there are 4 terms, you have to group them in pairs

Activity

Activity 1

Solve for θ (if necessary, correct to one decimal place)

1. $\tan 3\theta = \frac{-1}{\sqrt{3}}$
   $\theta \in [-60° ; 60°]$
2. $\tan(\theta + 20°) = \tan(130° - \theta)$
   $\theta \in [-180° ; 180°]$
3. $\tan\theta = -\tan 40°$
   $\theta \in [-360° ; 0°]$
4. $\sin \theta = 2 \cos \theta ; \theta \in [-180° ; 180°]$
5. $-\sin \theta = \frac{1}{2} \cos \theta$
   $\theta \in [-180° ; 180°]$
6. $\sin \theta + 3 \cos \theta = 0$
   $\theta \in [-90° ; 180°]$

Activity

Activity 2

Instruction: Find the general solutions to the following equations.

1. $\sin (3\theta + 24°) = -0{,}279$
2. $\cos (\theta + 50°) = -0{,}814$
3. $\tan^2 3\theta = 3$
4. $\cos 2x = \sin (x - 40)$
5. $\tan (90° - x) = \tan (2x + 60°)$
6. $\sqrt{2} \sin 2\theta = -1$

Activity

Activity 3

In each case, solve the equation according to the given domain.

1. $2 \sin^2\theta - 3 \cos^2\theta = 2$ if $\theta \in [-360° ; 360°]$
2. $2 \sin\theta \cos \theta - \sin \theta = 1 - 2 \cos \theta$
3. $\cos (5\theta - 40° = \sin (2\theta - 20°)$
4. $5 \sin^2\theta + 2 \cos \theta - 5 \cos \theta \sin \theta = 2 \sin \theta$

Activity

Activity 4

Solve:

1. $\cos 2A = -\cos 3A$
2. $2 \cos^2 x = 3 \sin x + 3$ if $x \in [-360° ; 360°]$
3. $10 \cos^2\theta - 10 \sin \theta \cos \theta + 2 = 0$
4. $\cos (\theta - 30°) = \sin 50°$
5. $\cos 3x = -\sin 4x$
6. $3 \sin^2 x - 5 \cos^2 x = 0$
7. $\cos^2 x - \sin^2 x = 3 \sin x + 2$
8. $\cos^2\theta - 1 = 2 \sin \theta \cos \theta$
9. $2 \sin x - \frac{\cos x}{\sin x} + 2 \cos x = 1$ if $x \in [-180° ; 360°]$