REDUCTION FORMULA

Learning Outcomes and Assessment Standards

Learning Outcome 3: Space, shape and measurement
Assessment Standard AS 3.5(c)
Derive the reduction formulae for:
$\sin(90^\circ \pm \alpha)$, $\cos(90^\circ \pm \alpha)$,
$\sin(180^\circ \pm \alpha)$, $\cos(180^\circ \pm \alpha)$, $\tan(180^\circ \pm \alpha)$,
$\sin(360^\circ \pm \alpha)$, $\cos(360^\circ \pm \alpha)$, $\tan(360^\circ \pm \alpha)$,
$\sin(-\alpha)$, $\cos(-\alpha)$, $\tan(-\alpha)$

Overview

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In this lesson you will:

- Learn to reduce all angles to co-terminal angles in the first quadrant
- Simplify trigonometric expressions by writing ratios in terms of $\sin \alpha$ and $\cos \alpha$
- Prove more trigonometric identities by examining the left-hand side and the right-hand side.

Lesson

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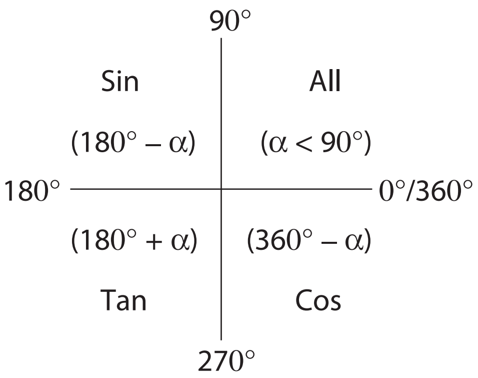
The horizontal reduction formulae:



Here we look at angles in terms of the horizontal line 180°/360°.

Remember that the CAST rule still applies in the quadrants.

So every angle will be reduced by this horizontal reduction formulae to an angle that lies in the first quadrant. We do this by looking at the CAST rule, and the size of the angle.

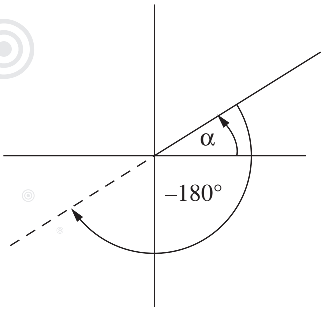
Let's try some:

- $\sin 125^\circ$ – (125° lies in the second quadrant)
  $= \sin(180^\circ - 55^\circ)$ – (the horizontal reduction formula in the 2nd Q)
  $= \sin 55^\circ$ (since the CAST rule says that sin is positive in 2nd Q)
- $\cos(180^\circ + \theta)$ – ($180^\circ + \theta$ lies in the 3rd Q and cos is negative here)
  $= -\cos\theta(180^\circ - \theta)$ – ($180^\circ - \theta$ in 2nd Q; tan negative here)
  $= -\tan\theta$
- $\sin(180^\circ - \theta)$ – ($180^\circ - \theta$ in 2nd Q; sin is positive here)
  $= \sin\theta$.

Thinking of negative angles:

How do we measure the angle $(\alpha - 180^\circ)$? Instead of learning them by rote, let us unpack them visually.

We know positive angles are measured anti-clockwise, and negative angles are measured clockwise. So $\alpha - 180^\circ$ will be: α = anti-clockwise, then -180° becomes 180° clockwise.



This tells us we are in the 3rd quadrant. Here tan is positive!

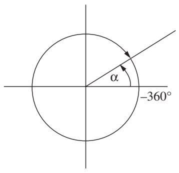
So $\sin(\alpha - 180°) = -\sin\alpha$

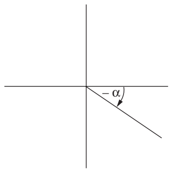
and $\cos(\alpha - 180°) = -\cos\alpha$

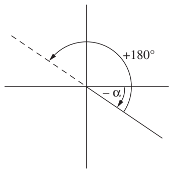
$\tan(\alpha - 180°) = \tan\alpha$

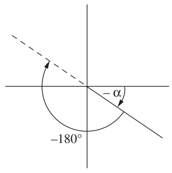
Let's try more:

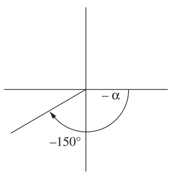
- $\tan 214° \rightarrow$ (214° is in 3rd Q: 180° + 34 by horizontal reduction)
  $= \tan(180° + 34°)$ (In 3rd Q tan is positive)
  $= \tan 34°$

- $\sin(\alpha - 360°) \rightarrow$  – which is in the first Q
  $= \sin\alpha$

- $\tan^2(-\alpha) \rightarrow$  – in the 4th Q: tan is negative
  $= (-\tan\alpha)^2 \rightarrow$ anything that is squared is positive
  $= \tan^2\alpha$

- $\cos(-\alpha + 180°) \rightarrow$  2nd Q; cos is negative
  $= -\cos\alpha$

- $\sin(-\alpha - 180°) \rightarrow$  2nd Q; sin is positive
  $= \sin\alpha$

- $\tan(-150°) \rightarrow$  3rd Q; reduction (180° + α);
  ; tan is negative
  ; 30° away from – 180°

  $= -\tan 30°$

Example 1

Simplify the following

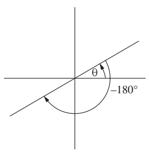
$$\frac{\cos(180° + \theta)\sin(\theta - 180°)}{\cos(-\theta)\tan(180° + \theta)}$$

Solution

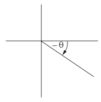
We look at one factor at a time to make sense of each one.

(1) $\cos(180° + \theta) \rightarrow$ (180° + θ in 3rd Q → here cos is negative)

$= -\cos\theta$

(2) $\sin(\theta - 180°) \rightarrow$  3rd Q where sin is negative

$= -\sin\theta$

(3) $\cos(-\theta) \rightarrow$  4th Q where cos is positive

$= \cos\theta$

(4) $\tan(180° + \theta) \rightarrow$ 3rd Q where tan is positive

So: $\frac{\cos(180 + \theta)\sin(\theta - 180°)}{\cos(-\theta)\tan(180° + \theta)} = \frac{(-\cos\theta)(-\sin\theta)}{(\cos\theta)(\tan\theta)}$

$= \frac{\sin\theta}{\frac{\sin\theta}{\cos\theta}}$ $\quad (\tan\theta = \frac{\sin\theta}{\cos\theta})$

$= \cos\theta$

Now do Activity 1 A on page 54.

REMEMBER

Ask yourself:

- Which quadrant am I in?
- What is the sign of the ratios in the quadrant?

Example

Example 2

Prove $\frac{\cos x}{\sin x} = \frac{2\sin^2(180° + x)}{2\tan(180° + x) + 2\sin(-x)\cos x}$

Solution

Solution

RHS $\frac{2\sin^2 x}{2\tan x + 2(-\sin x)(\cos x)}$

$= \frac{2\sin^2 x}{1} \div \left(\frac{2\sin x}{\cos x} - \frac{2\sin x\cos x}{1}\right)$

$= \frac{2\sin^2 x}{1} \div \frac{2\sin x - 2\sin x\cos^2 x}{\cos x}$

$= \frac{2\sin^2 x}{1} \times \frac{\cos x}{2\sin x(1 - \cos^2 x)}$

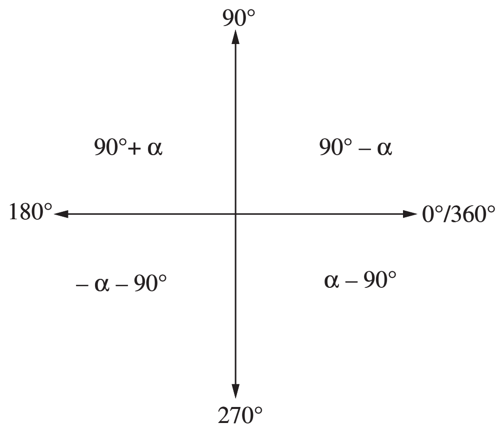
$= \frac{2\sin^2 x}{1} \div \frac{\cos x}{2\sin x \cdot \sin^2 x} = \frac{\cos x}{\sin x}$ =LHS

Now do Activity 1 B on page 54.

Vertical Reduction formulae

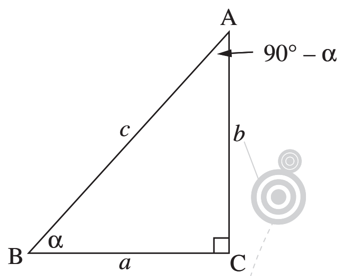
As the name suggests, we reduce angles in terms of the vertical line.

The CAST rule still applies here, but we now have to work with the complementary ratios.



Here is how they work:

In △ABC, $\hat{C} = 90°$ has been given. So $\hat{A} + \hat{B} = 90°$ since all angles in a triangle add to 180°. We call $\hat{A}$ and $\hat{B}$ complements of one another, or we say they are complimentary angles.



Also notice that $\sin\alpha = \frac{b}{c}$ $\quad = \cos(90° - \alpha)$

$\Rightarrow \sin\alpha = \cos(90° - \alpha)$

and $\cos\alpha = \frac{a}{c} = \sin(90° - \alpha) \Rightarrow \cos\alpha = \sin(90° - \alpha)$

Outside of the triangle we will see that for angles expressed as a vertical reduction:

sine becomes cosine and cosine becomes sine

So sin ↔ cos

Let's try some:

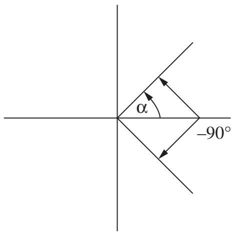
- sin(90° + α) → ((90° + α) in 2nd Q: sin is positive here; because of the 90° sin becomes cos.)

  = cos α

- cos(90° + α) → ((90° + α) in 2nd Q: cos is negative here; cos becomes sin because of 90°.)

  = sin α

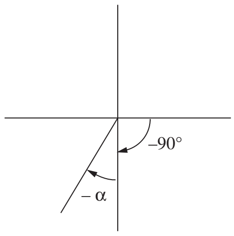
- cos(α – 90°) →



4th Quadrant; cos is positive here and cos becomes sin

  = sin α

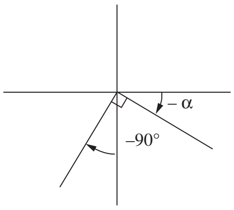
- cos(–90° – α) →



in the 3rd Q: cos is negative and cos becomes sin

  = –sin α

- sin(– α – 90°) →



3rd Q: sun is negative and sin → cos

  = cos α

Example

Example 3

Simplify $\frac{\tan(180° + x)\cos(90° - x)}{\sin(90° - x)} - \frac{\cos(180° - x)}{\sin(90° + x)}$

Again: tan (180° + x) → [3rd Q: tan positive] = tan x

cos (90° – x) → [1st Q: cos positive; cos → sin] = sin x

sin (90° – x) → [1st Q: sin positive; sin → cos] = cos x

cos (180° – x) → [2nd Q: cos negative] = – cos x

sin (90° + x) → [2nd Q: sin positive; sin → cos] = cos x

$= \frac{(\tan x)(\sin x)}{\cos x} - \frac{(-\cos x)}{\cos x}$

$= \frac{\sin x}{\cos x} \cdot \frac{\sin x}{\cos x} + 1$ $\qquad \left(\tan x = \frac{\sin x}{\cos x}\right)$

$= \frac{\sin^2 x + \cos^2 x}{\cos^2 x}$

$= \frac{1}{\cos^2 x}$ $\qquad (\sin^2 x + \cos^2 x = 1)$

Example 4

Prove that $\dfrac{\cos^2(90° - x) + \sin^2(90° + x)}{\tan(180° + x)\cdot\sin(x - 90°)} = -\dfrac{1}{\sin x}$

Solution

LHS $\dfrac{\sin^2 x + \cos^2 x}{(\tan x)(-\cos x)}$

$= \dfrac{1}{\left(\dfrac{\sin x}{\cancel{\cos x}}\right) \times \dfrac{(-\cancel{\cos} x)}{1}}$

$= \dfrac{1}{-\sin x}$ =RHS

Now do Activity 2A and 2B on page 54.

More complementary angles

If $\alpha + \beta = 90°$, we say α and β are complementary angles

we know $\sin \alpha = \cos (90° ° \alpha)$

so $\sin 20° = \cos 70°$

$\cos 40° = \sin 50°$

and $\dfrac{\cos 10°}{\sin 80°} = 1$ and $\dfrac{\sin 20°}{\cos 70°} = 1$

Example 5

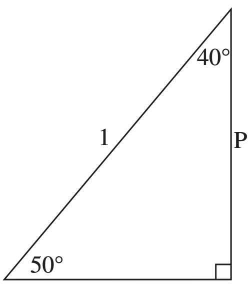
If $\sin 50° = p$, find in terms of p

a) $\cos 40°$ b) $\cos 50°$

Solution

Using a diagram

$\sin 50° = p$



According to Pythagoras $x = \sqrt{1 - p^2}$

So: (a) $\cos 40° = \dfrac{p}{1} = p$

(b) $\cos 50° = \dfrac{\sqrt{1 - p^2}}{1}$

Using identities

(a) $\cos 40° = \cos (90° - 50°)$

$= \sin 50°$

(b) $\cos 50°$:

$\cos^2 50° + \sin^2 50° = 1$

$\therefore \cos^2 50° = 1 - \sin^2 50°$

$\therefore \cos 50 = \sqrt{1 - p^2}$

Now do Activity 3 on page 54.

Activity 1

A. Simplify:

1) $1 - \dfrac{\sin^2(180° + A)}{1 - \cos(180° + A)}$

2) $\dfrac{1}{\tan(-\theta)\cos(180° - \theta)} - \dfrac{\cos^2(-\theta)}{\sin(180° - \theta)}$

3) $\dfrac{\tan A}{\sin(180° - A)} + \dfrac{\sin^2(-A}{\cos(180° + A)}$

4) $\dfrac{\sin(180° - \theta)\sin\theta - \cos^2(360° - \theta)}{\tan(180° + \theta) + \frac{1}{\tan(-\theta)}}$

B. Prove the following identities:

1) $2\cos(180° - x)\sin(-x) = \dfrac{2\tan(180° + x)}{1 + \tan^2(-x)}$

2) $\dfrac{1}{\cos(360° - \theta)} - \tan(180° - \theta) = \dfrac{\cos(360° - \theta)}{1 + \sin(-\theta)}$

Activity 2

A. Simplify

1) $\dfrac{\sin(360° - A)\tan(180° + A)\sin(90° - A)}{\cos(180° + A)\cos(90° - A)\tan(180° - A)}$

2) $\dfrac{\cos(90° + \alpha)\cos(-\alpha)\sin(-\alpha)}{\sin(\alpha - 90°)\tan(360° - \alpha)\cos\alpha}$

3) $\dfrac{\sin(180° - \beta)\tan(180° - \beta)\sin^2(90° - \beta)}{\cos(360° - \beta)\cos(\beta - 90°)\cos(-\beta)}$

B. Prove that:

1) $\dfrac{\cos^2(90° - x) + \sin^2(90° + x)}{\tan(180° + x)\sin(x - 90°)} = \dfrac{1}{\sin x}$

2) $\left(\dfrac{\cos\alpha\cos(180° + \alpha)}{\cos(90° - \alpha)}\right) = \dfrac{1}{\tan^2(-\alpha)} - \sin^2(90° + \alpha)$

3) $\dfrac{1}{\sin(180° - A) + 1} - \dfrac{1}{\cos(90° - A) - 1} = \dfrac{2}{\cos^2 A}$

Activity 3

1) Write the following at a ratio of 20°

a) sin 250° b) cos 340° c) tan 160°

2) If cos 35° = m, find in terms of m

a) sin 305° b) sin 245° c) cos 245°

3) If x and y are complementary angles and $2\cos x = \sqrt{2}$, find

a) $\cos^2 x ° \sin^2 y$ b) $\tan x \,.\, \tan y$

4) If A and B are supplementary angles and $\tan\beta = -\frac{8}{15}$, find sin A + cos B

5) Without a calculator, prove

a) $\dfrac{2\cos 80°}{\sin 10°} = 2$ b) $\dfrac{\cos 340°}{2\sin 110°} = \dfrac{1}{2}$

c) $\dfrac{\sin 70° \cos 175°}{\cos 340° \cos 185°} = 1$ d) $\dfrac{\sin^2 70° + \sin^2 20°}{\sin^2 40°(1 + \tan^2 50° = 1)}$