QUADRATIC AND EXPONENTIAL EQUATIONS

Using substitution methods

Learning Outcomes and Assessment Standards

Learning Outcome 1: Number and Number relationships
Learning Outcome 2: Functions and Algebra
Assessment Standard

- Expressions and equations using the Laws of exponents for rational exponents.
- Quadratic equations.

Overview

In this lesson you will:

- Use a substitute variable to help us solve a quadratic or exponential equation.
- Mix concepts that we have covered in previous lessons.
- Practise mathematical modelling.

Lesson

Example 1

$(2x^2 + x)^2 - 5(2x^2 + x) + 6 = 0$

similar brackets
powers differ

Solution

$\therefore$ Let $k = 2x^2 + x$

Then $k^2 - 5k + 6 = 0$

$\therefore (k - 3)(k - 2) = 0$

$\therefore k - 3 = 0$ or $k - 2 = 0$

But $k = 2x^2 + x$: Then $2x^2 + x - 3 = 0$ or $2x^2 + x - 2 = 0$

$\therefore (2x + 3)(x - 1) = 0$

$\therefore 2x = -3$ or $x = 1$

$\therefore x = -\frac{3}{2}$

Example 2

$x^2 - 3x + \frac{40}{x(x-3)} = 14$: Solve for x

$\Rightarrow x^2 - 3x + \frac{40}{x(x-3)} = 14$

$\therefore x(x - 3) + \frac{40}{x(x-3)} - 14 = 0$ Restrictions: $x \neq 0; x \neq 3$

Let $k = x(x - 3)$

Then $k + \frac{40}{k} - 14 = 0$

Now: $k^2 - 14k + 40 = 0$ ($\times k$ and get into standard form)

$\therefore (k - 10)(k - 4) = 0$

INDEPENDENT NEWSPAPERS

Now: $x(x-3)-10=0$ and $x(x-3)-4=0$

$\therefore x^2-3x-10=0$ $\quad x^2-3x-4=0$

$\therefore (x+2)(x-5)=0$ $\quad \therefore (x+1)(x-4)=0$

$\therefore x=-2$ or $x=5$ $\quad \therefore x=-1$ or $x=4$

3. Solve for x: $\sqrt{x^2-x-3}-5=-x^2+x$

Solution

Create a k. Isolate the surd.

$\sqrt{x^2-x-3}-5=-(x^2-x)$

$\sqrt{(x^2-x)-3}=-(x^2-x)+5$

Let $x^2-x=k$

$\sqrt{k-3}=-k+5$

Restriction: $k-3\geq 0 \Rightarrow k\geq 3$
$-k+5\geq 0 \Rightarrow k\leq 5$
$\therefore 3\leq k\leq 5$

$k-3=(5-k)^2$

$k-3=25-10k+k^2$

$0=k^2-11k+28$

$0=(k-7)(k-4)$

$k=7$ or $k=4$ Check

$k\neq 7$ since $3\leq k\leq 5$

$x^2-x=4$

$x^2-x-4=0$

Check: $b^2-4ac=(-1)^2-4(1)(-4)$ So $k=4$ is only the solution

$=1+16$

$=17$

So $x=\frac{1\pm\sqrt{17}}{2}$

Now do Activity 1.

Solving exponential equations using substitution

Example 4

Solve for x: $5^{x+1}-24=5^{1-x}$

Solution

Split up the exponents. Make the exponents positive.

$5^x{\cdot}5-24=5^1{\cdot}5^{-x}$

$5{\cdot}5^x-24=\frac{5}{5^x}$

let $5^x=k;\ k>0$

$5k-24=\frac{5}{k}$

$5k^2-24k=5$

$5k^2-24k-5=0$

$(5k+1)(k-5)=0$

$k\neq -\frac{1}{5}$ or $k=5\ (k>0)$

invalid $\quad 5^x=5$

$x=1$

Why is $k>0$? Since we are substituting 5^x with k, and we know that $5^x>0$ for all x, it will have the consequence that $k>0$ since $k=5^x$

Example 5

Solve for x: $3^x + 3^{4-2x} = 1 + 3^{4-x}$

Solution

$3^x + 3^{4-2x} = 1 + 3^{4-x}$

$\therefore 3^x + \frac{3^4}{3^{2x}} = 1 + \frac{3^4}{3^x}$ (Remember $a^{-m} = \frac{1}{a^m}$)

$\times 3^{2x}$: $3^{3x} + 3^4 = 3^{2x} + 3^x.3^4$

Let $3^x = k \rightarrow$ then $3^{2x} = k^2$ and $3^{3x} = k^3$

$\therefore k^3 - k^2 - 3^4k + 3^4 = 0$

$\therefore k^2(k - 1) - 3^4(k - 1) = 0$

$\therefore k^2 = 3^4$ or $k = 1$

$\therefore 3^{2x} = 3^4$ $\quad 3^x = 1 = 3^0$

$2x = 4$ $\quad \therefore x = 0$

$x = 2$

Now do Activity 2.

Mathematical modelling

1. Factorise $1 - \frac{1}{m^2}$
2. Hence, without using a calculator, evaluate
   $\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{16}\right)\left(1 - \frac{1}{25}\right) \ldots \left(1 - \frac{1}{400}\right)$

Solution

1. $\left(1 - \frac{1}{m}\right)\left(1 + \frac{1}{m}\right)$ Now use in the 2nd part of the question
2. $\left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\left(1 + \frac{1}{4}\right) \ldots \left(1 - \frac{1}{20}\right)\left(1 + \frac{1}{20}\right)$

   Work out each bracket

   $\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(\frac{4}{5}\right) \ldots \left(\frac{19}{20}\right)\left(\frac{21}{20}\right)$

   Notice that consecutive products cancel out one another

   $\left(\frac{1}{2}\right)\left(\frac{\cancel{3}}{\cancel{2}}\right)\left(\frac{\cancel{2}}{\cancel{3}}\right)\left(\frac{\cancel{4}}{\cancel{3}}\right)\left(\frac{\cancel{3}}{\cancel{4}}\right) \ldots \left(\frac{\cancel{20}}{\cancel{19}}\right)\left(\frac{\cancel{19}}{\cancel{20}}\right)\left(\frac{21}{20}\right)$

   Answer:

   $\left(\frac{1}{2}\right)\left(\frac{21}{20}\right) = \frac{21}{40}$

Activity 1

Solve for x:

1. $3x^2 - x + \frac{6}{3x^2 - x + 2} = 5$
2. $(x^2 - 5x)^2 - 2(x^2 - 5x) = 24$
3. $\sqrt{3x - 1} + 1 = \frac{6}{\sqrt{3x - 1}}$
4. $(x^2 - 3x)^2 - 5(x^2 - 3x) + 4 = 0$ (correct to 2 decimal places)
5. $4(3x^2 + x + 1) - \frac{10(3x^2 + x - 1)}{3x^2 + x} = 7$
6. $2x - 3 - \frac{3}{2x - 1} = 0$
7. $x^2 + 6x - 2 = \frac{35}{x^2 + 6x}$
8. $(2x^2 - 3x)^2 - 4x^2 + 6x - 3 = 0$
9. $\frac{\sqrt{2x - 1}}{2} - \frac{4}{\sqrt{2x - 1}} - 1 = 0$
10. $x^2 - \frac{5}{x^2 + x} = 4 - x$

Activity 2

Solve:

1. $3^{2x} - 12 \cdot 3^x + 27 = 0$
2. $3^{x+1} - 10 \cdot 3^{x-1} + 3 = 0$
3. $\left(2^{\frac{x}{2}} - 4 \cdot 2^{-\frac{x}{2}}\right)\left(2^{\frac{x}{2}} - 8 \cdot 2^{-\frac{x}{2}}\right) = 0$
4. $4 \cdot 2^{\frac{2x}{3}} - 33 + 8 \cdot 2^{-\frac{2x}{3}} = 0$
5. $4 \cdot 2^{2x+2} - 17 \cdot 2^x + 1 = 0$
6. $8 \cdot 2^{2x} + 10 \cdot 2^x - 3 = 0$