

# QUADRATIC EQUATIONS

*Completing the square*

## Learning Outcomes and Assessments Standards

### Learning Outcomes 2: Functions and algebra Assessment Standard

Solutions of a quadratic equation by completing the square and the derivation of the quadratic formula.

## Overview

In this lesson you will:

- Learn an algorithm to enable you to write a quadratic equation in the completed square format.
- Solve the equation.
- Decide when there are solutions and when there are no solutions.
- Use this technique to develop a formula to solve quadratic equations.

### *Lesson*

Solving equations by completing the square and developing a formula to solve quadratic equations.

Look at the following example:

#### Perfect square equation

$$(x - 3)^2 = 8$$

$$x - 3 = \pm\sqrt{8}$$

$$x = 3 \pm \sqrt{8}$$

It is sometimes asked to give the answer in simplest surd form:

$$\text{Then } \sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}$$

$$\text{So } x = 3 \pm 2\sqrt{2}$$

#### The non-perfect square equation

$$x^2 - 3x - 5 = 0$$

$$x^2 - 3x + \left(\frac{-3}{2}\right)^2 - \left(\frac{-3}{2}\right)^2 - 5 = 0$$

We add and subtract the square of half the number before  $x$  term and we get

$$x^2 - 3x + \left(\frac{-3}{2}\right)^2 - \frac{9}{4} - 5 = 0$$

Factorise the trinomial

$$\left(x - \frac{3}{2}\right)^2 = \frac{9}{4} + 5$$

Simplify

$$\left(x - \frac{3}{2}\right)^2 = \frac{9 + 20}{4}$$

$$x - \frac{3}{2} = \pm\sqrt{\frac{29}{4}}$$

The number before the  $x$  term is called the coefficient of  $x$

$$x = \frac{3 \pm \sqrt{29}}{2}$$

Note that this means that  $x$  is irrational since  $\sqrt{29}$  is an irrational number

Let's do another one

Solve by completing the square

$$2x^2 + x - 8 = 0$$

**Step 1: Divide by 2**

$$x^2 + \frac{1}{2}x - 4 = 0$$

**Step 2: Space out**

$$x^2 + \frac{1}{2}x - 4 = 0$$

**Step 3: Add and subtract the square of half the number before  $x$**

$$x^2 + \frac{1}{2}x + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - 4 = 0$$

If you are afraid of working with fractions, here is an easier method which uses the square completion to solve an equation.

- For  $ax^2 + bx + c = 0$
- (1) Multiply any term with  $4a$
  - (2) Add  $b^2$  to each side
  - (3) You solve.

Let's see:

- $x^2 - 3x - 5 = 0$      $a = 1$  so  $4a = 4$  and  $b^2 = (-3)^2 = 9$

$$4x^2 - 12x + 9 - 20 = 0 + 9$$
 [We multiplied each term with 4 and added 9 to each side]

$$\therefore (2x - 3)^2 = 29$$

$$\therefore 2x - 3 = \pm\sqrt{29}$$

No fractions

$$\therefore 2x = 3 \pm\sqrt{29}$$

$$x = \frac{3 \pm\sqrt{29}}{2}$$

- $2x^2 + x - 8 = 0$      $a = 2$  so  $4a = 4(2) = 8$  and  $b^2 = (1)^2 = 1$

$$\therefore 8(2x^2 + x - 8) + 1 = 0 + 1$$

$$\therefore 16x^2 + 8x - 64 + 1 = 1$$

$$\therefore 16x^2 + 8x + 1 = 65$$

$$\therefore (4x + 1)^2 = 65$$

$$4x + 1 = \pm\sqrt{65}$$

No fractions

$$4x = -1 \pm\sqrt{65}$$

$$x = \frac{-1 \pm\sqrt{65}}{4}$$

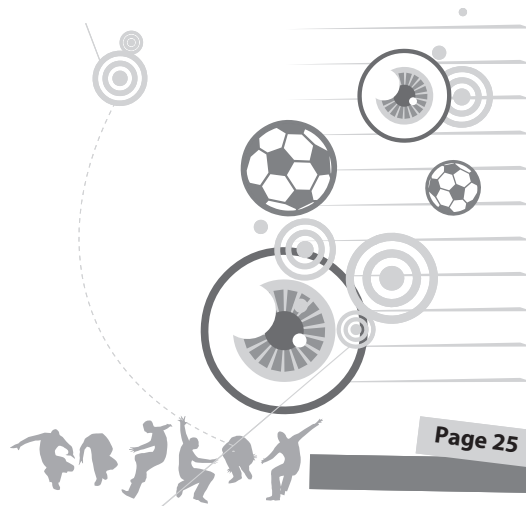
**Step 4: Leave the trinomial on its own**

$$x^2 + \frac{1}{2}x = \left(\frac{1}{4}\right)^2 = \frac{1}{16} + 4$$

**Step 5: Factorise the trinomial and simplify the right hand side**

$$\left(x + \frac{1}{4}\right)^2 = \frac{1 + 64}{16}$$

Why can we divide by 2?  
Because we are working with an equation (and not an expression) of which the one side is zero, so we can lose the two in the front.



Solve

$$x + \frac{1}{4} = \pm \frac{\sqrt{65}}{4}$$
$$x = -\frac{1}{4} \pm \frac{\sqrt{65}}{4}$$

### Careful of negatives

#### Solve by completing the square

##### Method 1

$$-3x^2 - 6x + 1 = 0$$

$$x^2 + 2x - \frac{1}{3} = 0$$

$$x^2 + 2x + (1)^2 - (1)^2 - \frac{1}{3} = 0$$

$$x^2 + 2x + (1)^2 = 1 + \frac{1}{3}$$

$$(x + 1)^2 = 1\frac{1}{3}$$

$$x + 1 = \pm \sqrt{\frac{4}{3}}$$

$$x = -1 \pm \frac{2}{\sqrt{3}}$$

$$x = \frac{-\sqrt{3} \pm 2}{\sqrt{3}}$$

##### Method 2

$$-3x^2 - 6x + 1 = 0 \quad 4(-3) = -12$$
$$b^2 = 36$$

$$\therefore -12(-3x^2 - 6x + 1) + 36 = 36$$

$$\therefore 36x^2 + 72x - 12 + 36 = 36$$

$$\therefore (36x^2 + 72x + 36) = 36 + 12$$

$$(6x + 6)^2 = 48$$

$$\therefore 6x + 6 = \pm \sqrt{48}$$

$$6x + 6 = \pm 4\sqrt{3}$$

$$x = \frac{+2(-3 \pm 2\sqrt{3})}{6}$$

$$x = \frac{-3 \pm 2\sqrt{3}}{3}$$

Now do Activity 1 Nos 1–7.

### General form

#### Solve by completing the square

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\left(x + \frac{b}{2a}\right) = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$ax^2 + bx + c = 0 \quad \times 4a$$

$$4a^2x^2 + 4abx + 4ac = 0 \quad \text{add } b^2$$

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac$$

$$\therefore (2ax + b)^2 = b^2 - 4ac$$

$$2ax + b = \pm \sqrt{b^2 - 4ac}$$

$$2ax = -b \pm \sqrt{b^2 - 4ac}$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

### Formula to solve quadratic equations

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

so

$$\text{Solve: } 3x^2 - 4x - 5 = 0$$

$$a = 3 \quad b = -4 \quad c = -5$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{4 \pm \sqrt{16 - 4(3)(-5)}}{6}$$

$$x = \frac{4 \pm \sqrt{16 + 60}}{6}$$

$$x = \frac{4 \pm \sqrt{76}}{6}$$

Now do Activity 1 Nos 8–9.

### Activity 1

Solve by completing the square

1.  $x^2 - 2x - 7 = 0$
2.  $x^2 - 4x + 7 = 0$
3.  $p^2 - 3p + 1 = 0$
4.  $2x^2 - 3x - 1 = 0$
5.  $x^2 - 2x - 24 = 0$
6.  $5x^2 - 3x - 1 = 0$
7.  $5 - x - 2x^2 = 0$
8.  $3x^2 - 6x - 2 = 0$
9.  $9x^2 - 6mx + m^2 - 3m = 0$

