LESSON 3

NUMBER AND NUMBER RELATIONSHIPS

Exponents

Learning Outcomes and Assessment Standards

Learning Outcome 1: Number and number relationships
Asssessment Standards
We know this when the learner is able to:
- Simplify expressions using the laws of exponents;
- Simplify expressions for rational exponents; and
- Add, subtract, multiply and divide simple surds.

Overview

In this lesson you will:

- Explore laws of exponents covered in Grade 10.
- Explore laws of exponents involving rational exponents and surds.
- Simplify expressions involving exponents and surds.
- Complete activities in the form of worksheets for practice.
- Complete an investigation for your portfolio.

Lesson

Laws of exponents

1. **Product Law**: $a^m \times a^n = a^{m+n}$
2. **Quotient Law:** $\frac{a^m}{a^n} = a^{m-n}$

From the Quotient Law we get:

a) Negative exponents

$\frac{a^3}{a^4} = a^{-1} = \frac{1}{a}$ so

$4p^{-1}q^2 = \frac{4q^2}{p}$ and

$3m^{-1} = \frac{3}{m}$ and

$\frac{6}{x^{-2}} = 6x^2$ and

$a + b^{-1} = a + \frac{1}{b}$ and

$(x+3)^{-2} = \frac{1}{(x+3)^2}$

and $(2m)^{-2} = \frac{1}{(2m)^2} = \frac{1}{(4m)^2}$

b) The exponent zero

$\frac{a^4}{a^4} = a^0 = 1$ so $6^o = 1$

$3p^0 = 3$ but $(3p)^0 = 1$

$3 + m^0$ but $(3+m)^0$

$= 3 + 1$ $=1$

$= 4$

$a + b^0$ but $(a+b)^0$

$= a + 1$ $= 1$

The Power Law

Example of using the Power Law

$(a^m)^n = a^{mn}$

$(a^m p^n)^q = a^{mq} \cdot p^{nq}$

$\left(\frac{a^m}{b^n}\right)^p = \frac{a^{mp}}{b^{nq}}$

Example

$(2m^2n^{-3}p)^{-2}$

$= 2^{-2}\cdot m^{-4}\cdot n^6\cdot p^{-2}$

$= \frac{n^6}{4m^4p^2}$

Rational exponents and roots

$\sqrt[n]{m} = m^{\frac{1}{n}}$ $\qquad \sqrt[n]{a^m} = a^{\frac{m}{n}}$

so in general

$\sqrt{p} = p^{\frac{1}{2}}$ $\qquad \sqrt[3]{m} = m^{\frac{1}{3}}$ $\qquad \sqrt[3]{a^2} = a^{\frac{2}{3}}$ $\qquad \sqrt{m^5} = m^{\frac{5}{2}}$

From Rational Exponents and Roots

1. $4^{\frac{1}{2}} = (2^2)^{\frac{1}{2}} = 2$
2. $\sqrt[3]{8} = 8^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2$
3. $(27)^{\frac{2}{3}} = (3^3)^{\frac{2}{3}} = 3^2 = 9$
4. $(8)^{-\frac{2}{3}} = (2^3)^{-\frac{2}{3}} = 2^{-2} = \frac{1}{4}$
5. $\sqrt[4]{16^3} = (16)^{\frac{3}{4}} = (2^4)^{\frac{3}{4}} = 8$
6. $\left(\frac{8}{27}\right)^{-\frac{2}{3}} = \left(\frac{2^3}{3^3}\right)^{-\frac{2}{3}} = \frac{2^{-2}}{3^{-2}} = \frac{9}{4}$

Watch the negative sign

1. $-9^{\frac{1}{2}} = -(32)^{\frac{1}{2}} = -3$
2. $(-9)^{\frac{1}{2}}$ is invalid. Why?
3. $-8^{\frac{1}{3}} = -(2^3)^{\frac{1}{3}} = -2$
4. $(-8)^{\frac{1}{3}} = (-2^3)^{\frac{1}{3}} = -2$ Why?
5. $(-27)^{\frac{2}{3}} = (-3^3)^{\frac{2}{3}} = (-3)^2 = +9$
6. $(-9)^{\frac{3}{2}}$ is invalid
7. $(-27)^{-\frac{2}{3}} = (-3^3)^{-\frac{2}{3}} = (-3)^{-2} = \frac{1}{9}$
8. $(-16)^{-\frac{3}{2}}$ is invalid

We can't find the square root of a negative (this applies to all even roots).

Example 1

Simplify:

$\frac{6^{x+1}\cdot 4^{x-1}\cdot 3^{x-2}}{12^{x+1}\cdot 2^x\cdot 3^{x-1}}$

$= \frac{(3^1\cdot 2^1)^{x+1}\cdot (2^2)^{x-1}\cdot 3^{x-2}}{(2^2\cdot 3)^{x+1}\cdot 2^x\cdot 3^{x-1}}$

$= \frac{3^{x+1}\cdot 2^{x+1}\cdot 2^{2x-2}\cdot 3^{x-2}}{2^{2x+2}\cdot 3^{x+1}\cdot 2^x\cdot 3^{x-1}}$

$= 3^{x+1+x-2-x-1-x+1}\cdot 2^{x+1+2x-2-2x-2-x}$

$= 3^{-1}\cdot 2^{-3}$

$= \frac{1}{3} \times \frac{1}{8} = \frac{1}{24}$

Note that this expression does not contain any terms!

So we:

(a) prime the bases

(b) collect like bases

Example 2

$\frac{16\cdot 8^n}{4^{n+1}\cdot 2^{n+4}}$

$= \frac{2^4\cdot (2^3)^n}{(2^2)^{n+1}\cdot 2^{n+4}}$ Prime every base

$= \frac{2^4\cdot 2^{3n}}{2^{2n+2}\cdot 2^{n+4}}$ Drop the brackets

$= 2^{4+3n-2n-2-n-4}$ Collect powers

$= 2^{-2}$ Add/subtract powers

$= \frac{1}{2^2}$

$= \frac{1}{4}$

If no terms present:

(a) prime the bases

(b) collect like bases

INDEPENDENT NEWSPAPERS

Activity 1

Simplify.

1. $\left(\frac{27a^4}{8a}\right)^{\frac{2}{3}}$

2. $\left(\frac{16x^{-3}y^4}{9xy^{-2}}\right)^{-\frac{1}{2}}$

3. $\frac{10^{n+3}.5^{n-1}}{50^{n+2}}$

4. $\frac{9^{2k+1}.3^{-k}.4^{2-k}}{3^{5k-2}.6^{3-2k}.2}$

5. $\frac{3^{3n-1}.9^{-4n}.27^{n+1}}{3^{-(3n+1)}}$

6. $\frac{3^x.6^{2x-1}.2^{4x+1}}{12^{3x}}$

Activity 2

Example of addition and subtraction with exponents.

Rule Break down and factorise

Example:

1. $\frac{3{\cdot}2^{x-2} - 2^{x-1}}{2^x + 3{\cdot}2^{x-3}}$

$= \frac{3{\cdot}2^x{\cdot}2^{-2} - 2^x{\cdot}2^{-1}}{2^x + 3{\cdot}2^x{\cdot}2^{-3}}$ Common factor

$= \frac{2^x(3{\cdot}2^{-2} - 2^{-1})}{2^x(1 + 3{\cdot}2^{-3})}$

$= \frac{\not{2^x}\left(\frac{3}{4} - \frac{1}{2}\right)}{\not{2^x}\left(1 + \frac{3}{8}\right)}$

Multiply each term by LCD = 8

$= \frac{6 - 4}{8 + 3}$

$= \frac{2}{11}$

Watch out! We have terms, so we

(a) prime the bases

(b) search for a common factor and factorise

(c) simplify by cancelling the common factors

Simplify:

1. $\frac{3^x + 3^{x+1}}{3^x - 3^{x+2}}$

2. $\frac{2{\cdot}2^{x+1} + 8{\cdot}2^{x-3}}{4{\cdot}2^{x-1} - 16{\cdot}2^{x-4}}$

3. $\frac{6{\cdot}3^{x-1} - 2{\cdot}3^{x+1}}{5{\cdot}3^x + 3^{x+2}}$

4. $\frac{5^x + 5^{x-2}}{2{\cdot}5^{x-1} - 3{\cdot}5^{x-2}}$

5. $\frac{(a - a^{-1})^2{\cdot}(a - 1)^{-1}}{a^{-2} + 2a^{-1} + 1}$

$= \frac{\left(a - \frac{1}{a}\right)^2}{(a - 1)^1{\cdot}\left(\frac{1}{a^2} + \frac{2}{a} + 1\right)}$ Remember: $a^{-m} = \frac{1}{a^m}$

$= \frac{\left(\frac{a^2 - 1}{a^2}\right)}{(a - 1)\left(\frac{a^2 + 2a + 1}{a^2}\right)}$ Add inside brackets with LCD

$= \frac{(a - 1)(a + 1)a^2}{(a - 1)(a + 1)^2a^2}$ Simplify

$= \frac{1}{a + 1}$

INVESTIGATION 1

Real numbers are divided into rational numbers and irrational numbers.

Rational numbers can be written as $\frac{a}{b}$ where *a* and *b* are both integers.

Examples are $\frac{3}{4}$; 4; 1,3; $1\frac{1}{4}$; $2,\dot{1}\dot{4}$.

We know exactly where to put rational numbers on the number line.

Irrational numbers

Cannot be written as $\frac{a}{b}$ where *a* and *b* are both integers.

Examples are

$\sqrt{2}$; $\sqrt{5}$; $\sqrt[3]{9}$; π. We know that $\sqrt{2}$ is somewhere between 1 and 2 – closer to 1 than 2, but we are not sure where to put $\sqrt{2}$ on the number line.

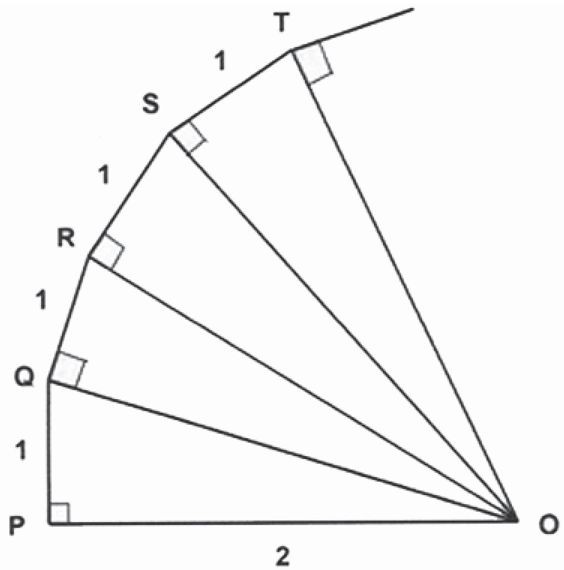
a) Can you use a number line and a construction to put $\sqrt{2}$ accurately on the number line. How about $\sqrt{5}$; $\sqrt{13}$; $\sqrt{20}$; $\sqrt{34}$; $\sqrt{45}$?

b) What common property do the above surds have? Make up some more.

c) Now use what you discovered above to put $\sqrt{3}$ $\sqrt{6}$ $\sqrt{7}$; (in fact, any surd) accurately on the number line by construction. (Approach your teacher if you need help).

INVESTIGATION 2

A sequence of triangles OPQ, OQR, ORS . . . is formed as shown below.



Calculate the lengths of OQ; OR; OS; OT (leave your answer in simplest surd form).

Write in terms of n, the hypotenuse of the nth triangle.

Investigate for different lengths of PO.