ALGEBRA

Complete the square for expressions

Learning Outcomes and Assesments Standards

Learning Outcomes 2: Algebra
Assessment Standard
Manipulate algebraic expressions by completing the square

Overview

In this lesson you will:

- Look for patterns in square binomials.
- Learn to complete the square of quadratic binomials expressions.
- Use this concept to find maximum or minimum values.
- Prove that expressions are positive or negative.
- Apply this concept in problem solving.

Lesson

Squares are useful – why?

We know that x^2 is always positive, and when $x = 0$, $x^2 = 0$.

So we write: $x^2 \geq 0$ for all x.

So if we have $(x + 2)^2$, then it will be zero if $x = -2$, but if $x \neq -2$, then it is always positive.

We write $(x + 2)^2 \geq 0$ for all x.

Now what happens when we have $(x + 2)^2 + 4$?

Let's see: $(x + 2)^2 \geq 0$ for all x.

If I add 4 both sides: $(x + 2)^2 + 4 \geq 0 + 4$

So we write: $(x + 2)^2 + 4 \geq 4$ which tells us that it is bigger or equal to 4. So minimum value is 4.

So if I have $(x - 1)^2 - 4$,
then $(x - 1)^2 \geq 0$
so $(x - 1)^2 - 4 \geq -4$
Here the minimum value is -4

- So why are squares useful? They guarantee that the term is positive!
  Also remember: if $(x + 2)^2 \geq 0$, then
  $-(x + 2)^2 \leq 0$
  So $-(x + 2)^2 + 4 \leq 4$
- From $y = ax^2 \pm bx + c$ to $y = a(x \pm p)^2 \pm q$
  If $ax^2 \pm bx + c = a(x \pm p)^2 \pm q$
  Then $ax^2 \pm bx + c = a(x^2 - 2px + p^2) + q$
  $= ax^2 - 2apx + ap^2 + q$
  Divide by a: $x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 - 2px + p^2 + \frac{q}{a}$

Comparing the coefficient:

For x: $\frac{b}{a} = -2p \Rightarrow p = -\frac{b}{2a}$

constant: $\frac{c}{a} = p^2 + \frac{q}{a}$

$$c = ap^2 + q \Rightarrow q = c - ap^2$$

Look at the following squares and find a pattern.

$(x + 3)^2$	$(x - 1)^2$
$= x^2 + 6x + (3)^2$	$= x^2 - 2x + (-1)^2$
$(x + 4)^2$	$\left(x - \frac{1}{2}\right)^2$
$= x^2 + 8x + (4)^2$	$= x^2 - x + \left(-\frac{1}{2}\right)^2$

What pattern do you see?

The third term of the trinomial is (half of the co-efficient of x term)^2.

Examples

What must be added to the following expressions so that the expressions will result in a perfect square?

1) $x^2 + 6x$ 2) $x^2 + 5x$ 3) $x^2 - 4x$

4) $x^2 - 3x$ 5) $x^2 + \frac{1}{2}x$ 6) $x^2 - \frac{2}{3}x$

Solutions

1. $x^2 + 6x + \left(\frac{6}{2}\right)^2$
$= x^2 + 6x + 9$

2. $x^2 + 5x + \left(\frac{5}{2}\right)^2$
$= x^2 + 5x + \frac{25}{4}$

3. $x^2 \left(\frac{-4}{2}\right)^2$
$= x^2 - 4x + 4$

4. $x^2 - 3x + \left(\frac{-3}{2}\right)^2$
$= x^2 - 3x + \frac{9}{4}$

5. $x^2 + \frac{1}{2}x + \left(\frac{\frac{1}{2}}{2}\right)^2$
$= x^2 + \frac{1}{2}x + \left(\frac{1}{2} \times \frac{1}{2}\right)^2$
$= x^2 + \frac{1}{2}x + \left(\frac{1}{4}\right)^2$
$= x^2 + \frac{1}{2}x + \frac{1}{16}$

6. $x^2 - \frac{2}{3}x + \left(\frac{-\frac{2}{3}}{2}\right)^2$
$= x^2 - \frac{2}{3}x + \left(-\frac{1}{3}\right)^2$
$= x^2 - \frac{2}{3}x + \frac{1}{9}$

Now do Activity 1.

Completing the square

Examples

Complete the square for the following expressions:

1. $x^2 - 4x + 10$
$= x^2 - 4x + (-2)^2 - (-2)^2 + 10$
$= x^2 - 4x + 4 - 4 + 10$
$= (x - 2)^2 + 6$

Conclude with $\underbrace{x^2 - 4x + 10}_{\text{form } ax^2 + bx + c} = \underbrace{(x - 2)^2 + 6}_{\text{form } a(x - p)^2 + q}$

The expression is always positive

The minimum value is 6 when $x = 2$

2. $-x^2 + 3x + 1$

Solution

Take out a factor of –1

$-(x^2 - 3 - 1)$

Add and subtract $\left(-\frac{3}{2}\right)^2$

$= -\left[x^2 - 3x + \left(-\frac{3}{2}\right)^2 - \frac{9}{4} - 1\right]$

$= -\left[\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} - 1\right]$

$= -\left(x - \frac{3}{2}\right)^2 + \frac{13}{4}$

The expression has a maximum value of $\frac{13}{4}$ when $x = \frac{3}{2}$

3. $2x^2 - 3x - 1 = 2\left[x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2 - \frac{1}{2}\right]$

$= 2\left[\left(x - \frac{3}{4}\right)^2 - \frac{9}{16} - \frac{8}{16}\right]$

$= 2\left(x - \frac{3}{4}\right)^2 - \frac{17}{8}$

$\therefore x^2 + 3x + 1 = -\left(x - \frac{3}{2}\right)^2 + \frac{13}{4}$

There is an easier way.

$2x^2 - 3x - 1$

$= [16x^2 - 24x - 8]\frac{1}{8}$ $\quad \times 4(2) = 8$ and $\div 8$

$= [16x^2 - 24x + 9 - 9 - 8]\frac{1}{8}$ add $(-3)^2$ and subtract $(-3)^2$

$= [(4x - 3)^2 - 17]\frac{1}{8}$

Practise exercise (work with a friend)

Complete the square for the following expressions. Attempt the exercise first and then check you answers.

1. $x^2 + 4x + 2$ 2. $2x^2 - 3x + 6$ 3. $-2x^2 + 4x - 5$

Solutions

1. $x^2 + 4x + 2$

$= x^2 + 4x + (2)^2 + 2$ $\left[\text{add and subtract } \left(\frac{4}{2}\right)^2\right]$

$= (x + 2)^2 - 4 + 2$ (form the square)

$= (x + 2)^2 - 2$

The minimum value of the expression is –2.

2. $2x^2 - 3x + 6$

Take out a common factor of 2

$2\left(x^2 - \frac{3}{2}x + 3\right)$

Add $\left(-\frac{3}{4}\right)^2$ and subtract $\left(-\frac{3}{4}\right)^2$ in the brackets.

Factorise the quadratic trinomial

$2[\left(x-\frac{3}{4}\right)^2-\frac{9}{16}+3]$

Distribute

$2\left(x-\frac{3}{4}\right)^2-\frac{9}{8}+6$

$=2\left(x-\frac{3}{4}\right)^2+\frac{39}{8}$

Conclusions

$2\left(x-\frac{3}{4}\right)^2+\frac{39}{8}$ tells us that the expression is always positive.

The expression has a minimum value of $\frac{39}{8}$ when $x=\frac{3}{4}$.

3. $-2x^2+4x-5$

Solution

$-2\left[x^2-2x+\frac{5}{2}\right]$

$=-2\left[x^2-2x+(-1)^2-1+\frac{5}{2}\right]$

$=-2\left[(x-1)^2+\frac{3}{2}\right]$

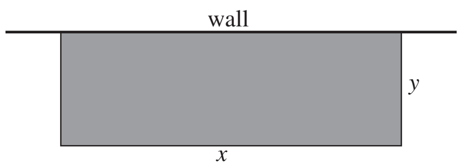
$=-2(x-1)^2-3$

- the expression has a maximum value of –3 when $x = 1$.
- The expression is always negative.

Now do Activity 2 numbers 1 to 2

Mathematical modelling

James intends to build a rectangular enclosure for his chickens. He has 100 m of fencing. He intends using a wall for one side of the enclosure.



a) Formulate an expression in x for the area of the enclosure.

b) What will the maximum area of this enclosure be?

c) Which dimensions gives this maximum area?

a) $2y+x=100$

$2y=100-x$

$y=50-\frac{1}{2}x$

$A=x\left(50-\frac{1}{2}x\right)$

$A=-\frac{1}{2}x^2+50x$

b) $A=-\frac{1}{2}[x^2-100x+(-50)^2-2500]$

$A=-\frac{1}{2}(x-50)^2+1250$

Max. area 1 250 m^2

Dimensions

c) $x = 50$ $y = 25$

Now do Activity 2 no 4.

Activity 1

What term must be added to each of the following expressions so that the expression will result in a perfect square.

1. $x^2 + 7x$
2. $x^2 - 6x$
3. $x^2 + \frac{1}{4}x$
4. $x^2 - \frac{2}{5}x$

Activity 2

1. Complete the square of each of the following expressions and write the expression in the form $a(x \pm p)^2 \pm q$

   Say whether the expression has a maximum or minimum value and determine the max or min value.

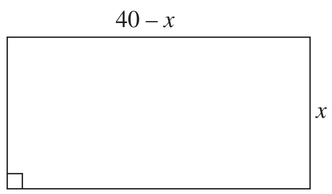
   a) $x^2 + 3x - 7$
   b) $-x^2 - 5x - 1$
   c) $3x^2 + 2x + 7$
   d) $5x^2 + 4x + 3$
   e) $x^2 + px + 3$
   f) $px^2 + qx + r$

2. Prove that the following quadratic expressions are always positive.

   a) $x^2 - x + 5$
   b) $2x^2 - 3x + 8$
   c) $10x^2 + 5x + 2$

3. Prove that the following quadratic expressions are always negative.

   a) $-x^2 + 3x - 5$
   b) $-4x^2 - 2x - 1$

4. 

   i) Find an expression for the area of the above rectangle.
   ii) Find the maximum area of the rectangle.
   iii) What are the dimensions of the rectangle so that the area is a maximum.