learning channel

STUDY Mate

Department of Education

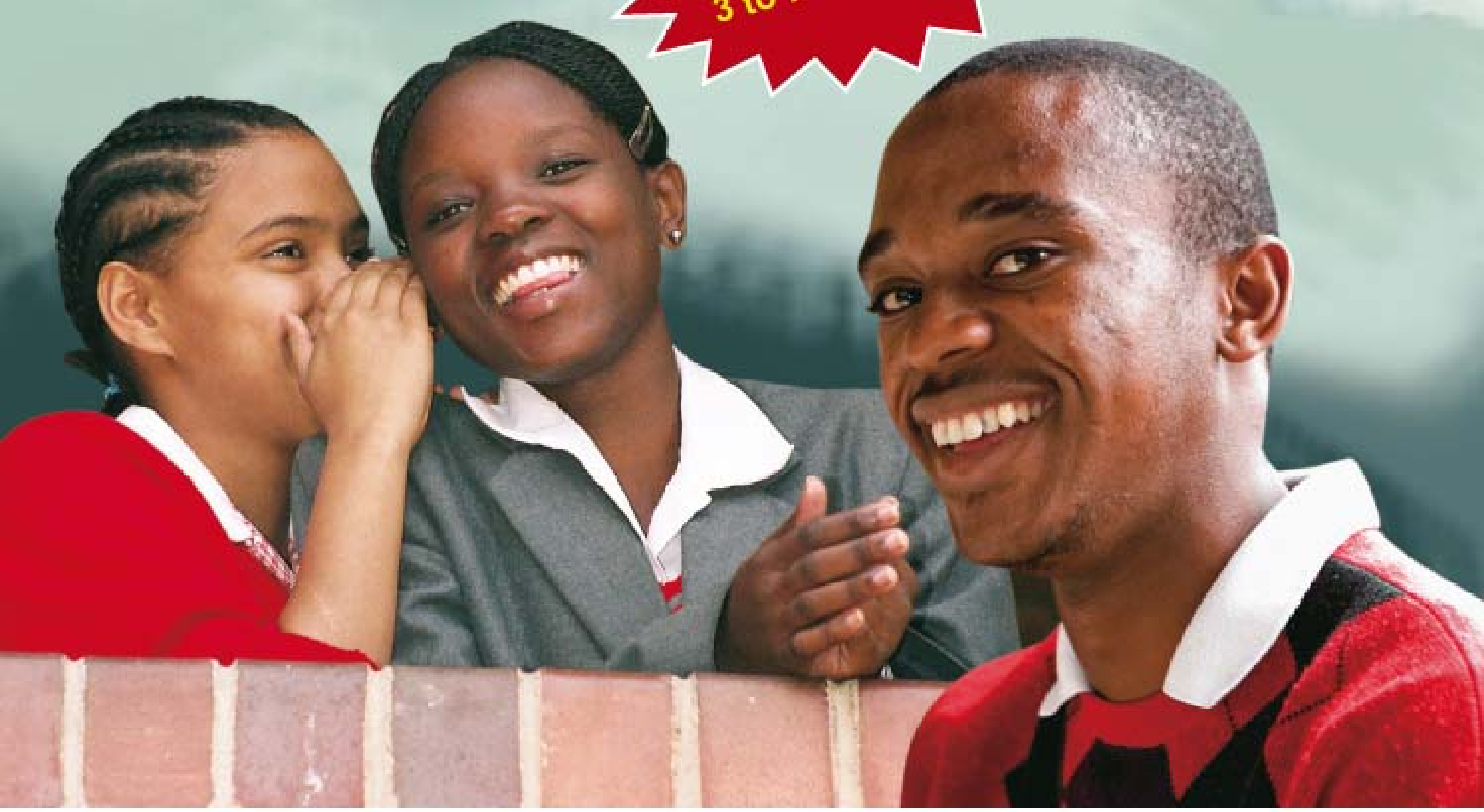
GRADES 8–11 ISSUE 10 SEPTEMBER 2007

Learning activities for Grades 8–11


Education Minister
Naledi Pandor

Content

SPECIAL EDITION
Maths Week:
3 to 7 September



Dear Learner

The week of 3 to 7 September 2007 is celebrated as Mathematics Week nationally. Mathematics is one of the most important learning areas at school because, as you know, maths forms the basis for many scientific and technological skills that our country so desperately needs. For this reason, and in support of Mathematics Week, we have decided to fill this issue of Study Mate with lots of maths and maths literacy activities. We would also like to draw your attention to one of the most exciting events of the maths world, namely the National Maths Olympiad. We will be publishing examples of Maths Olympiad questions in the next few weeks. While not all of you will be participating in the Olympiad, all of you can be doing some extra maths to prepare yourselves not only for your exams, but for all those things in life that require some basic mathematical skills. And, by the way, did you know that maths can be entertaining too? Look out in your local newspaper for the Sudoku puzzle - you can have hours of fun with that and grow those brain cells at the same time. Happy Maths Week - get into the swing of it and you are sure to reap the rewards!

A proud collaboration between the Department of Education and The South African Maths Foundation

Mathematical Literacy Lesson Notes

The advantages of buying in bulk

As part of an entrepreneurial project some students are investigating the possibility of making cookies and selling these. They have two recipes and established the costs of the various ingredients at their local supermarket.

Butter Cookies – ingredients	Sugar cookies – ingredients
250 g butter	$\frac{2}{3}$ cup margarine
375 g sugar	$\frac{1}{3}$ cup sugar
360 g flour	1 teaspoon baking powder
15 ml baking powder	1 egg
a pinch of salt	1 teaspoon vanilla essence
2 large eggs	2 cups flour
5 ml vanilla essence	Makes 24 cookies
Makes 36 cookies	

Ingredient costs at one supermarket in Oct '05				
Butter	250 g	500 g		
	R11,49	R17,99		
Margarine	250 g	500 g	1 kg	
	R3,69	R7,49	R15,95	
Sugar	500 g	1 kg	2,5 kg	5 kg
	R6,58	R5,95	R13,89	R27,65
Flour	1 kg	2,5 kg	5 kg	10 kg
	R5,89	R8,99	R21,99	R38,99
Baking powder	50 g	100 g	200 g	
	R2,89	R7,59	R10,99	
Eggs	6 eggs	18 eggs	30 eggs	
	R5,95	R12,99	R21,45	
Vanilla essence	100 ml	500 ml		
	R5,99	20,99		
Salt	500 g	1 kg		
	R1,85	R3,59		

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Approximate Volume/Mass Conversion chart		Approximate mass in grams per 250 ml (1 cup) of ingredients	
250 ml	1 cup	Flour	120 g
190 ml	$\frac{3}{4}$ cup	Butter	230 g
125 ml	$\frac{1}{2}$ cup	Margarine	215 g
60 ml	$\frac{1}{4}$ cup	Sugar	200 g
15 ml	1 tablespoon		
5 ml	1 teaspoon		

1. For each of the ingredients on the list provided determine the unit cost and complete the table below.

Butter	250 g	500 g		
	R11,49	R17,99		
Unit cost (R/kg)				
Margarine	250 g	500 g	1 kg	
	R3,69	R7,49	R15,95	
Unit cost (R/kg)				
Sugar	500 g	1 kg	2,5 kg	5 kg
	R6,58	R5,95	R13,89	R27,65
Unit cost (R/kg)				
Flour	1 kg	2,5 kg	5 kg	10 kg
	R5,89	R8,99	R21,99	R38,99
Unit cost (R/kg)				
Baking powder	50 g	100 g	200 g	
	R2,89	R7,59	R10,99	
Unit cost (R/kg)				
Eggs	6 eggs	18 eggs	30 eggs	
	R5,95	R12,99	R21,45	
Unit cost (R/egg)				
Vanilla essence	100 ml	500 ml		
	R5,99	20,99		
Unit cost (R/l)				
Salt	500 g	1 kg		
	R1,85	R3,59		
Unit cost (R/kg)				

2. For each product on the table circle the option that gives the best value for money.

   When buying ingredients from the store, you have to buy them in whole units – for example although the recipe for butter cookies only requires 2 eggs, you cannot buy only 2 eggs. The store sells these in cartons of 6, 18 and 30 eggs and so you must buy either 6, 18 or 30 eggs. Although the 18 egg carton is better value for money at R0,72/egg than the 6 egg carton at R0,99/egg – the 6 egg carton is still the best option as it will cost less (R5,95 vs. R12,99) if you only need 2 eggs!
3. Calculate the minimum cost of buying enough ingredients to make a single batch (36 cookies) of butter cookies.
4. Using the conversion table provided convert the cup measures and teaspoons used in the Sugar cookie recipe to the units in which the ingredients are sold.
5. Calculate the minimum cost of buying enough ingredients to make a single batch (24 cookies) of sugar cookies.

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Mathematical Literacy Lesson Notes

If the students are to make money from their project they will have to make many batches of cookies. Not only will they make more cookies to sell but they will also enjoy the benefits that come from buying in bulk. Firstly there will be less wastage and secondly they will be able to buy in larger quantities that give better value for money

6. Determine the minimum cost of buying enough ingredients to make 3 batches (3 × 36 cookies) of butter cookies.
7. What is the cost/batch of butter cookies if you buy ingredients for 3 batches of cookies? How does this compare to the cost/batch of butter cookies if you buy ingredients for 1 batch?
8. Determine the minimum cost of buying enough ingredients to make 3 batches (3 × 36 cookies) of sugar cookies.
9. What is the cost/batch of sugar cookies if you buy ingredients for 3 batches of cookies? How does this compare to the cost/batch of sugar cookies if you buy ingredients for 1 batch?

   Your calculations should have revealed that the cost/batch of cookies decreased as you increased the amount of ingredients to be bought from 1 to 3 batches.

10. Investigate the impact on the cost/batch of cookies of increasing the number of batches by doing the necessary calculations and completing the table below

	Number of batches	1	2	3	5	8	10	12	15
Butter Cookies	Total cost								
	Cost/ batch								
Sugar Cookies	Total cost								
	Cost/ batch								

11. Discuss the trend that you observe in the table.

Mathematical Literacy Solutions

The advantages of buying in bulk

1. Unit cost in bold.
2. Best value for money is shown by a circle.

Butter	250 g	500 g		
	R11,49	R17,99		
Unit cost (R/kg)	**R45,96**	**R35,98**		
Margarine	250 g	500 g	1 kg	
	R3,69	R7,49	R15,95	
Unit cost (R/kg)	**R14,76**	**R14,98**	**R15,95**	
Sugar	500 g	1 kg	2,5 kg	5 kg
	R6,58	R5,95	R13,89	R27,65
Unit cost (R/kg)	**R13,16**	**R5,95**	**R5,56**	**R5,53**
Flour	1 kg	2,5 kg	5 kg	10 kg
	R5,89	R8,99	R21,99	R38,99
Unit cost (R/kg)	**R5,89**	**R3,60**	**R4,40**	**R3,90**
Baking powder	50 g	100 g	200 g	
	R2,89	R7,59	R10,99	
Unit cost (R/kg)	**R57,80**	**R75,90**	**R73,45**	
Eggs	6 eggs	18 eggs	30 eggs	
	R5,95	R12,99	R21,45	
Unit cost (R/egg)	**R0,99**	**R0,72**	**R0,89**	
Vanilla essence	100 ml	500 ml		
	R5,99	20,99		
Unit cost (R/l)	**R59,90**	**R41,98**		
Salt	500 g	1 kg		
	R1,85	R3,59		
Unit cost (R/kg)	**R3,70**	**R3,59**		

Mathematical Literacy Solutions (cont.)

3. Single batch (36 cookies) of butter cookies:

Ingredients	Amount needed	Quantity bought	Price
Butter	250g	250g	R11,49
Sugar	375g	1kg	R5,95
Flour	360g	1kg	R5,89
Baking Powder	15g	50g	R2,89
Salt	1g	500g	R2,59
Eggs	2	6	R5,95
Vanilla Essence	5ml	100ml	R5,99
		TOTAL	R40,75

4. Sugar cookies:

Ingredients	Amount needed	Conversion
Margarine	23 cup	150g
Sugar	13 cup	70g
Flour	2 cups	240g
Baking Powder	1 teaspoon	5ml
Eggs	1	1
Vanilla Essence	1 teaspoon	5ml

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Mathematical Literacy Solutions (cont.)

5. Single batch (24 cookies) of sugar cookies:

Ingredients	Amount needed	Quantity bought	Price
Margarine	150g	250g	R3,69
Sugar	70g	1kg	R5,95
Flour	240g	1kg	R5,89
Baking Powder	5 ml	50 g	R2,89
Eggs	1	6	R5,95
Vanilla Essence	5 ml	100 ml	R5,99
		TOTAL	R30,36

6. 3 batches (3 × 36 cookies) of butter cookies:

Ingredients	Amount needed	Quantity bought	Price
Butter	750 g	250 g 500 g	R11,49 R17,99
Sugar	1,125 g	2 × 1 kg	R11,90
Flour	1 080 g	2,5 kg	R8,99
Baking Powder	45 g	50 g	R2,89
Salt	3 g	500 g	R2,59
Eggs	6	6	R5,95
Vanilla Essence	15 ml	100 ml	R5,99
		TOTAL	R40,75

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Mathematical Literacy Solutions (cont.)

7. The unit cost of three batches is R22,60 per batch which is practically a half of what a single batch costs.
8. 3 batches (3×36 cookies) of sugar cookies:

Ingredients	Amount needed	Quantity bought	Price
Margarine	450g	2×250g	R7,38
Sugar	210g	1kg	R5,95
Flour	720g	1kg	R5,89
Baking Powder	15ml	50g	R2,89
Eggs	3	6	R5,95
Vanilla Essence	15ml	100ml	R5,99
		TOTAL	R34,05

9. The unit cost of three batches is R11,35 per batch which is practically a third of what a single batch costs
10.

	Number of batches	1	2	3	5	8	10	12	15
Butter Cookies	Total cost	R40,75	R47,25	R67,79	R96,43	R136,74	R172,35	R192,73	R244,88
	Cost/batch	R40,75	R23,63	R22,60	R19,29	R17,09	R17,24	R16,06	R16,33
Sugar Cookies	Total cost	R30,36	R34,05	R34,05	R40,84	R54,17	R57,86	R75,83	R89,66
	Cost/batch	R30,36	R17,03	R11,35	R8,17	R6,77	R5,79	R6,32	R5,98

11. Initially there is an enormous decrease in the unit price of a batch of cookies as the number of batches increases. The unit price does; however, seem to stabilise after about 8 batches.

Recursive Sequences

Prof W Olivier, NMMU

Let's get into some serious maths

Introducing learners to the idea of recursive sequences.

Objectives:

Upon completion of this lesson, learners will:

1. have been introduced to recursive number sequences
2. have learned the terminology used with recursive sequences
3. have experimented with creating different basic recursive sequences by varying the starting number(s) and the rules to generate new numbers from existing ones
4. have practised determining the starting values that should be used to produce a desired sequence
5. have seen and worked with both the explicit (closed) form and the recursive form of a number sequence

Key terms

number sequence	An ordered set of real numbers whose elements are usually determined based on some function of the counting numbers.
recursion	Given some starting (initial) information and a rule for how to apply it to get new information, the rule is then repeated using the new information.
Iteration	A number of rules or steps that are repeated over and over again.

The terminology of recursion (recursively-defined sequences, recursive description, recursive function, recurrence relations, etc.) is quite confusing at first. Many confuse these ideas with iteration, a general term for describing repetitive algorithms.

1. Focus and Review

Remember what has been learned in previous lessons that will be pertinent to this lesson. Think specifically about the general "rule" that arithmetic and geometric sequences of numbers must satisfy.

2. Preview

This lesson will work with arithmetic and geometric sequences, their recursive and explicit formulas and finding terms in a sequence.

3. Implementation Strategies

Consider first a few examples of familiar Arithmetic Sequences and Geometric Sequences.

Example 1

3, 6, 9, 12, 15, 18, ...

Recursive form:

$$a_1 = 3;\ a_n = a_{n-1} + 3$$

$\uparrow$ a_n only known after $n - 1$ steps

In this form, whatever term we have, we are able to find the term that immediately follows it. We can find the sequence itself if we also have the starting (initial) term. In a recursive sequence the 'next' term depends on the 'now' term.

Explicit (closed) form: $a_n = 3 + 3(n - 1)$

This can be recognized as an arithmetic sequence, usually expressed as $a_n = a_1 + (n - 1)d$.

Each term of the sequence can be found provided we know what the first term (a_1) is, and what the common difference (d) is.

The explicit (closed) form can be obtained for any given arithmetic or geometric sequence. For example:

(1) 0, 3, 6, 9, ...

Explicit form: $a_n = 0 + 3(n - 1)$

This is an arithmetic sequence, almost the same as the one above, except that it starts at 0 instead of at 3.

(2) 3, 9, 27, ...

Explicit form: $a_n = a_1 r^{n-1}$

This is a geometric sequence, which starts at 3 ($a_1 = 3$); the common ratio (r) is also 3.

Example 2

In (2) above, if we choose $a_1 = 2$, we obtain a different sequence:

2, 6, 18, 54, 162, ...

The recursive form of this sequence is:

$$a_1 = 2;\ a_n = 3 \times a_{n-1}$$

$\uparrow$ a_n only known after $n - 1$ steps

The explicit form of this sequence is:

$$a_n = 3r^{n-1}$$

where r is the constant ratio

We need to know a_1 and r in order to find the terms in this sequence.

From (1) and (2) we see that we can generate infinitely many similar sequences, depending on the initial (starting) value. The examples also show the advantage of the closed form of a number sequence over the recursive form. One can directly calculate any term of a sequence from the closed form. In contrast, the n^{th} term only follows after a number of steps when the recursive form is used.

Exercise 1

Discover recursive rules of the following sequences for yourselves:

(a) $-2, 0, 2, 4, \ldots$

(b) $1, 5, 9, \ldots$

(c) $\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \ldots$

(d) $10, 20, 30, \ldots$

(e) $-1, 2, -4, 8, \ldots$

(f) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

Example 3

In this example we have a combination of an arithmetic and a geometric sequence:

1, 3, 7, 15, 31, ...

We can express this in the recursive form as:

$$a_1 = 1;\ a_n = 2 \times a_{n-1} + 1$$

$\uparrow$ a_n only known after $n - 1$ steps

How can this be expressed in the explicit form?

When we investigate these terms we find that:

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Recursive Sequences

$a_1 = 1 = 2^0$

$a_2 = 3 = 2^0 + 2^1$

$a_3 = 7 = 2^0 + 2^1 + 2^2$

$a_4 = 15 = 2^0 + 2^1 + 2^2 + 2^3$

$a_5 = 31 = 2^0 + 2^1 + 2^2 + 2^3 + 2^4$

From the pattern that has evolved we can express this as follows:

$a_n = 2^0 + 2^1 + 2^2 + 2^3 + \dots 2^{n-1}$.

Example 4

The previous example showed a linear recursive number sequence of order one. In this example we have a linear recursive number sequence of order two. It is a well-known sequence, named after the mathematician Leonardo Fibonacci, who lived in the first half of the thirteenth century. The Fibonacci sequence is

1, 1, 2, 3, 5, 8, 13, ...

It is very difficult to give the closed form of this sequence. Its recursive form is:

$a_1 = 1;\ a_2 = 1;\ a_n = a_{n-1} + a_{n-2}$

$\uparrow \quad n \geq 3$

a_n only known after $n - 1$ steps

Please visit the following website on Fibonacci Numbers: (http://www.world-mysteries.com/sci_17.htm)

Practical example

The linear function $y = 50x + 230$, which describes the line with y-intercept 230 and slope 50, can be used to represent the distance from home (y) of a traveller who began the day 230 kilometres from home and drives at 50 kilometres per hour for x hours. Defined recursively, this relationship would be

Next = Now + 50, where Start = 230

The terms of this sequence are thus

230, 280, 330, 380, 430, ...

Exercises

1. Find the recursive and explicit form of the sequence:
   0,5; 0,05; 0,005; 0,0005; ...
2. Find the recursive and explicit form of the sequence:
   5; 10; 20; 40; ...
3. Find the sequence with the same recursive rule as in 2., but with a starting value of –2. What do you notice?
4. Give the first four terms of the sequence defined by the closed form:
   $a_1 = 2;\ a_n = a_{n-1}^2;\ n \geq 2$. Challenge: Find the explicit form of the sequence.
5. Find the 6th term of the sequence defined by the recursive form:
   $a_1 = 2;\ a_2 = 1;\ a_n = a_{n-1} + a_{n-2};\ n \geq 3$.

Solutions

1. Recursive form: $a_1 = 0{,}5;\ a_n = a_{n-1} \div 10$
   Explicit form: $a_n = 0{,}5\left(\frac{1}{10}\right)^{n-1}$
2. Recursive form: $a_1 = 5;\ a_n = 2a_{n-1}$
   Explicit form: $a_n = 5(2)^{n-1}$
3. Sequence:
   $-2, -4, -8, \dots$
4. First four terms of sequence are:
   2, 4, 16, 256
5. $a_6 = 8$ (This sequence gives rise to the well-known Lucas series.)

Linear Programming

R Fray & B May

1. Historical background

Problems seeking to maximize (or minimize) a numerical function of a number of variables, subject to certain constraints, form a general class known as optimization problems. These problems were first encountered in the physical sciences and led to the development of such techniques as differential calculus and the calculus of variations. More recently new optimization problems have emerged in the field of economics for which the above techniques were found to be of limited use. These problems are known as programming problems and are concerned with determining optimal allocations of limited resources to meet given objectives. More specifically, they are concerned with situations where a number of resources (eg personnel, materials, land, etc) are combined to yield one or more products. There are, however, restrictions on the total amount of each resource, the total amount of products required, the quality of the products, etc. The problem is to determine that allocation of resources which will either minimize the costs involved, or alternatively maximize the profit.

Linear programming problems are programming problems in which all relationships between variables are linear. The general linear programming problem was first formulated and solved by George B. Dantzig in 1947. His method of solution is known as the simplex algorithm.

2. Example of a linear programming problem

Consider a factory which has three types of machines, Machines A, B and C, and which turns out four products, Products 1, 2, 3 and 4. Each of the products has to undergo some sort of processing on each machine (in the same sequence of activity).

The table below shows the processing time per product per machine, the unit profit per product and the total time available per machine per week. For example, Machine type A requires 1,5 hours to produce 1 unit of Product 1. We wish to determine the weekly output for each product in order to maximize profits.

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Machine type	Products				Total time available
	1	2	3	4	
A	1,5	1	2,4	1	2 000
B	1	5	1	3,5	8 000
C	1,5	3	3,5	1	5 000
Unit profit	5,24	7,30	8,34	4,18	

Suppose x_j represents the number of units produced of product j. The value of x_j cannot be increased arbitrarily, because machine time is limited.

Thus, the machines of type A are in use for:

$(1{,}5x_1 + 1x_2 + 2{,}4x_3 + 1x_4)$ hours per week,

so that we must have:

$1{,}5x_1 + x_2 + 2{,}4x_3 + x_4 \leq 2\,000$ ... (3.1)

(It would be wrong to use only an = sign in the above inequality, because there might not be any combination of production rates which would use each machine type to capacity, and we do not wish to predict which machines are to be used to capacity.)

Similarly, we must have for type B and C machines, respectively:

$x_1 + 5x_2 + x_3 + 3{,}5x_4 \leq 8\,000$ ... (3.2)

$1{,}5x_1 + 3x_2 + 3{,}5x_3 + x_4 \leq 5\,000$ ... (3.3)

Since, furthermore, we cannot produce negative quantities of any product, we must have:

$x_j \geq 0, \quad j = 1, 2, 3, 4$ ... (3.4)

The total profit per week to be maximized is z, where :

$z = 5{,}24x_1 + 7{,}30x_2 + 8{,}34x_3 + 4{,}18x_4$ ... (3.5)

The above is clearly a programming problem, and furthermore it is linear, since all relationships (3.1) to (3.5) are linear.

Inequalities (3.1) to (3.3) are known as the **constraints.**

Inequality (3.4) is known as the **non-negativity restriction**.

Equation (3.5) is known as the **objective function.**

It is worth noting that even for the above very simple case the problem is non-trivial (and hence will not be solved here) and the optimal product mix is far from obvious.

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Linear Programming

3. Graphical methods

Simple Linear Programming problems (LPPs) involving only two variables can easily be solved by a graphical method. The geometrical (graphical) interpretation of LPPs with two variables is quite important and provides a great deal of insight into the structure of problems with more variables. The following examples are given to develop a greater understanding of the graphs of inequalities, and the relevant optimization techniques.

3.1 Examples

3.1.1 Maximize $z = 5x + 3y$ subject to the following constraints:

$$3x + 5y \leq 15$$
$$5x + 2y \leq 10$$
$$y \geq 0$$
$$x \geq 0$$

These constraints become, respectively:

$$y \leq -\frac{3}{5}x + 3$$
$$y \leq -\frac{5}{2}x + 5$$
$$y \geq 0$$
$$x \geq 0$$

The feasible region (i.e. the set of points in the plane satisfying the constraints and the non-negativity restrictions) is the shaded (see comment below) region below:

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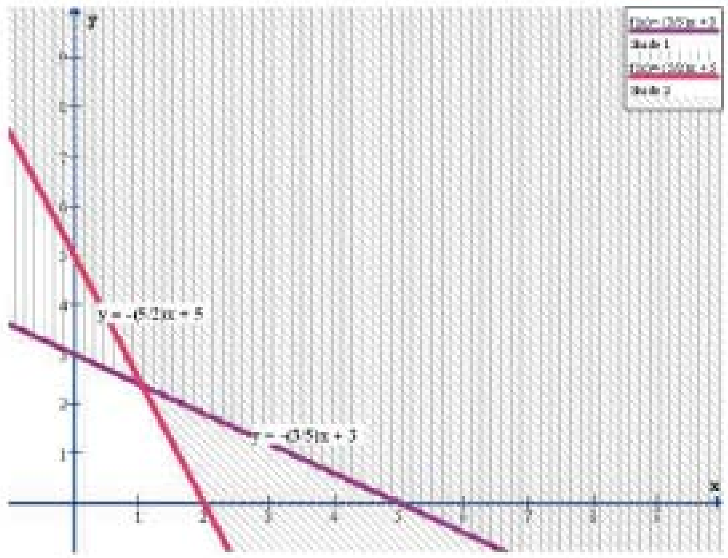

Figure 3.1

The authors elected to use the unshaded region in the first quadrant to represent the feasible region. For DoE purposes this should be drawn in the more usual way, with the shading **below** the relevant lines for the two constraints, **above** the x-axis, and to the **right** of the y-axis. This applies to later graphs as well.

Any point in the feasible region is known as a **feasible solution**.

The feasible region is bounded by:

(i) the two co-ordinate axes (the non-negativity restrictions ensure that all feasible solutions must lie in the first quadrant)

(ii) the line $3x + 5y = 15$ (the first constraint $3x + 5y \leq 15$ ensures that the feasible region is on or below the line $y \leq -\frac{3}{5}x + 3$) the line $5x + 2y = 10$ (the second constraint $5x + 2y = 10$ ensures that the feasible region is on or below the line $y \leq -\frac{5}{2}x + 5$).

Now, for each value of z, the line $z = 5x + 3y$ is a straight line. It can be rewritten as $y = -\frac{5}{3}x + \frac{z}{3}$. Allowing z to vary generates a set of parallel straight lines (z increases when these lines are shifted to the upper right as indicated by the broken lines in Figure 3.2).

It is now immediately obvious that the optimal solution to the problem is given by the upper right corner of the feasible region. This is the point where the two lines $3x + 5y = 15$ and $5x + 2y = 10$ intersect. Using a calculator we find that this point has the coordinates $y = 1{,}053$ and $x = 2{,}368$. The associated value of z is 12,37.

Once the shading in Figure 3.1 has been changed then in this figure perhaps only the feasible region needs to be shaded.

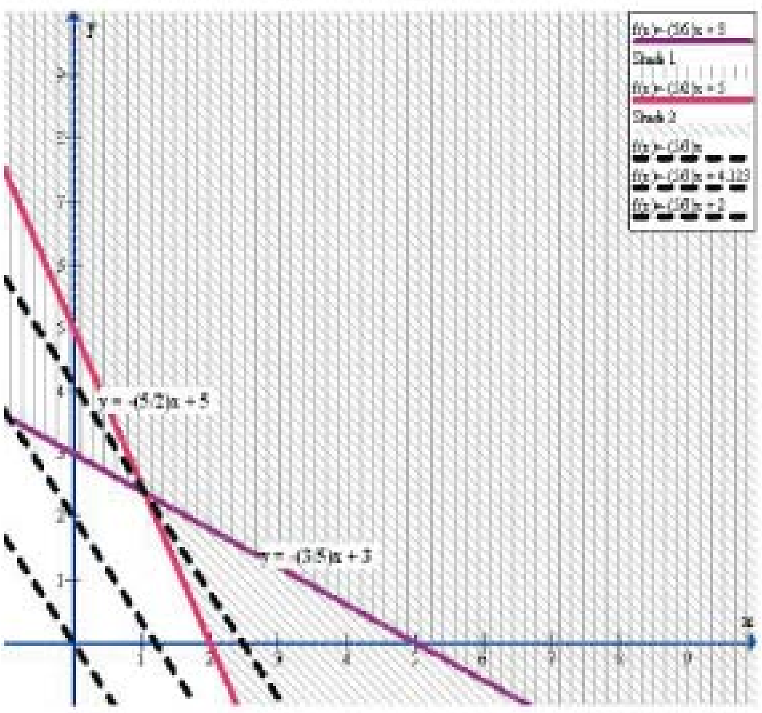

Figure 3.2

3.1.2 The same problem as 4.1.1, but with the objective function changed to $z = 5x + 2y$, gives the following:

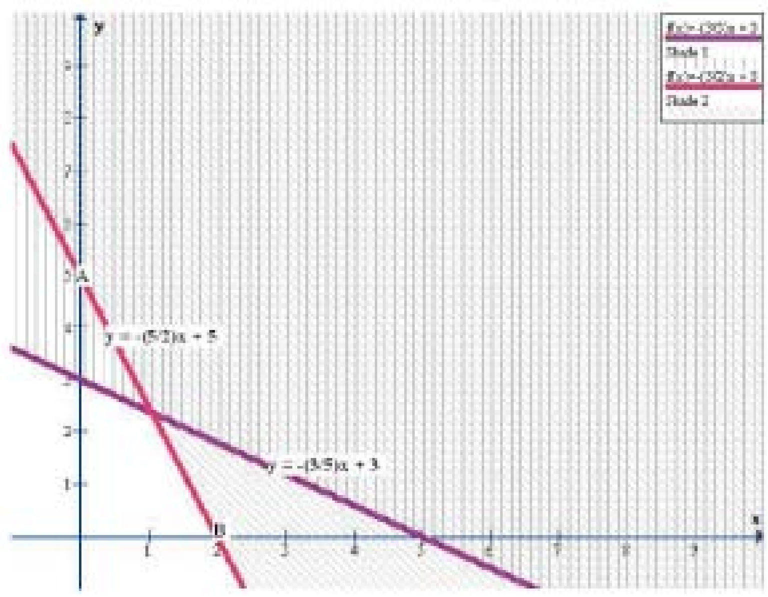

Figure 3.3

The lines representing $z = 5x + 2y$ are now parallel to one of the edges of the feasible region since both the objective function and one of the constraints have the same slope. This means that there is no longer a unique solution, but that the entire line segment from A to B gives optimal solutions. Note however that as in 3.1.1 at least one corner of the feasible region gives an optimal solution.

The two examples above illustrate normal (well-behaved) LPPs. The next two will illustrate two possible exceptional cases.

3.1.3 Maximize $z = 2x + 2y$ subject to the following constraints:

$$x - y \geq -1$$
$$-\frac{1}{2}x + y \leq 2$$
$$y \geq 0$$
$$x \geq 0$$

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Linear Programming

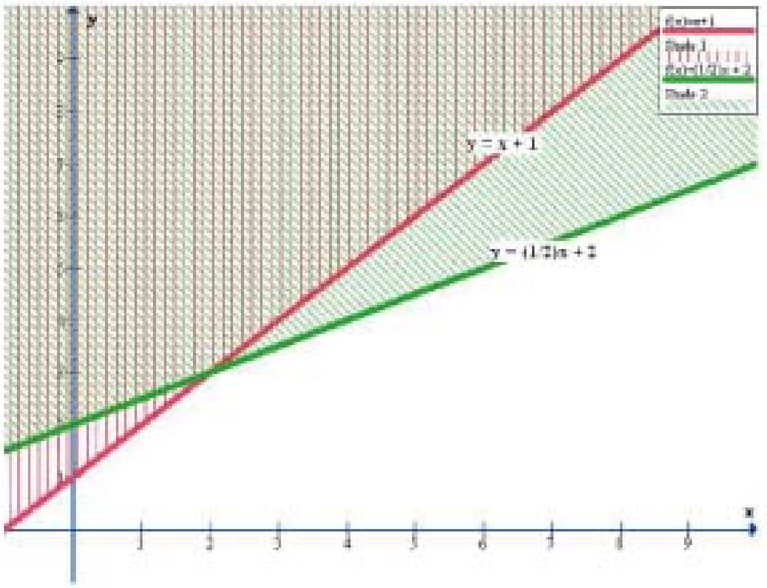

Figure 3.4

In this case the feasible region is unbounded i.e. there are many feasible solutions This is indicated by the unshaded area. See earlier comment re shading. NB: The little triangle above the x-axis and to the left of the y-axis should be shaded according to the current shading convention, since $x \geq 0$.

3.1.4 Maximize $z = 3x - 2y$ subject to the following constraints:

$$x + y \leq 1$$
$$2x + 2y \geq 4$$
$$y \geq 0$$
$$x \geq 0$$

These constraints become:

$$y \leq -x + 1$$
$$y \geq -x = 2$$
$$y \geq 0$$
$$x \geq 0$$

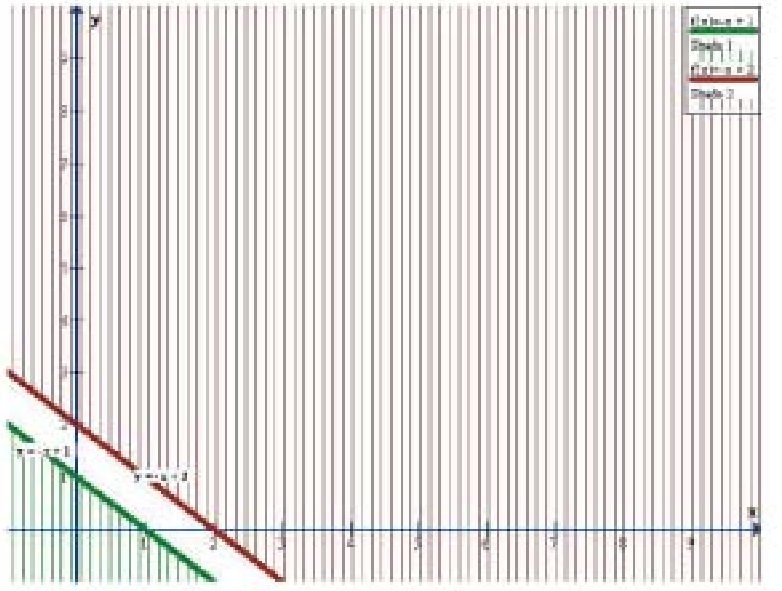

Figure 3.5

There is no point satisfying the constraints and therefore we say the problem is infeasible. (This is indicated by the fact that the shaded areas on the graph do not overlap).

NOTE: The occurrence of unbounded regions and infeasible solutions are in practice usually due to incorrect formulation of the problem.

From these few examples we can make the following generalisations:

(i) The feasible region (if there is one) is always a convex set (i.e. for any two points in the feasible region, the entire line segment joining these points lies in the feasible region).

(ii) Whenever the optimal value of z is finite, a corner of the feasible region will give the optimal solution.

It can be proved that these two observations are true in general.

The above examples also illustrate important facts concerning linear programming problems. These facts can be summarized as follows:

(i) A linear programming (LP) problem is the problem of maximizing or minimizing a linear function subject to a finite number of linear constraints.

(ii) The linear function that is to be maximized or minimized in a LP problem is called the **objective function.**

(iii) Numbers that satisfy all the constraints of a LP problem constitute a **feasible solution** of that problem.

(iv) A feasible solution that maximizes or minimizes the objective function is called an **optimal solution**. The corresponding value of the objective function is called the **optimal value** of the problem.

(v) Not every LP problem has a unique optimal solution. Some problems have many different solutions and others have no optimal solution at all.

(vi) Problems that have no feasible solutions are called **infeasible**.

(vii) Problems that have many feasible solutions are called **unbounded.**

(viii) A LP problem is said to be in **standard form** when all the constraints are linear inequalities and all variables are non-negative.

4. Prior knowledge required for Linear Programming

(a) Knowledge of linear equations:

- (i) solution of linear equations
- (ii) standard form of linear equations
- (iii) graphs of linear equations
- (iv) algebraic solutions of simultaneous linear equations

(b) Knowledge of linear inequalities:

- (i) solution of linear inequalities

5.1 Revision Exercises

(i) Revise linear equations and sketch graphs of these functions (start with the equation in non-standard form; write in standard form and then point out gradient, y-intercept and x-intercept; use intercept method to draw graphs indicating positive and negative slopes; find the point of intersection of line graphs using the algebraic method).

Example 1

Write the following equations in standard form and then draw the graph that represents each equation:

(a) $4 - 2y - x = 0$ (negative slope)

(b) $2x - 3y + 6 = 0$ (positive slope)

Solution

(a) $y = -\frac{1}{2}x + 2$

(b) $y = \frac{2}{3}x + 2$

The point of intersection of the two lines, i.e. the simultaneous solution of (a) and (b), is $x = 0$ and $y = 2$.

When these graphs are re-drawn it would be preferable if the headings EXAMPLE 1 etc. are deleted.

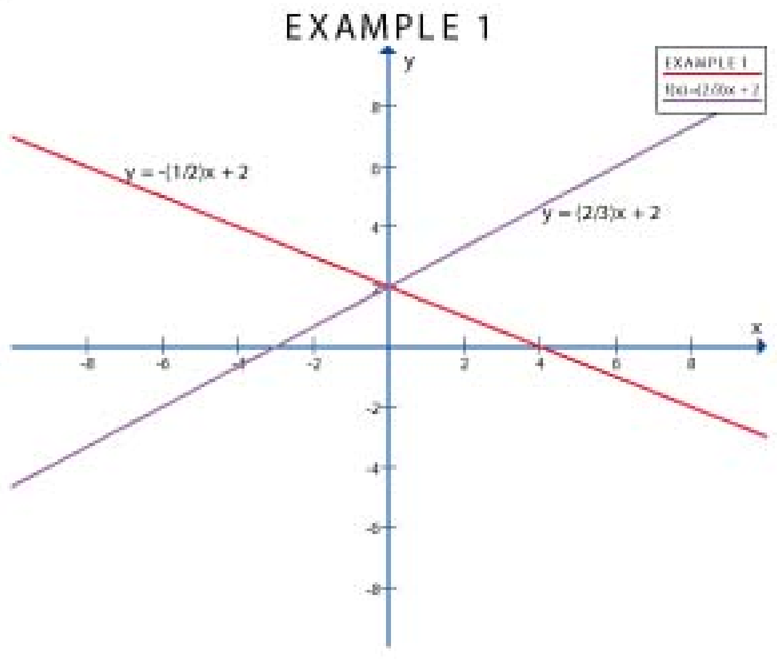

Figure 5.1

(ii) Revise linear inequalities and sketch graphs of inequalities:

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Linear Programming

Example 2

Draw the graph of:

(a) $y - 1 = x$

(b) $y + 2x = 1$

(c) $y = -4$

Then shade in your diagram to show the region representing ~~the graph of~~.

(d) $y - 1 > x$

(e) $y + 2x \leq 1$

(f) $y \geq -4$

Solution

(a), (b), (c)

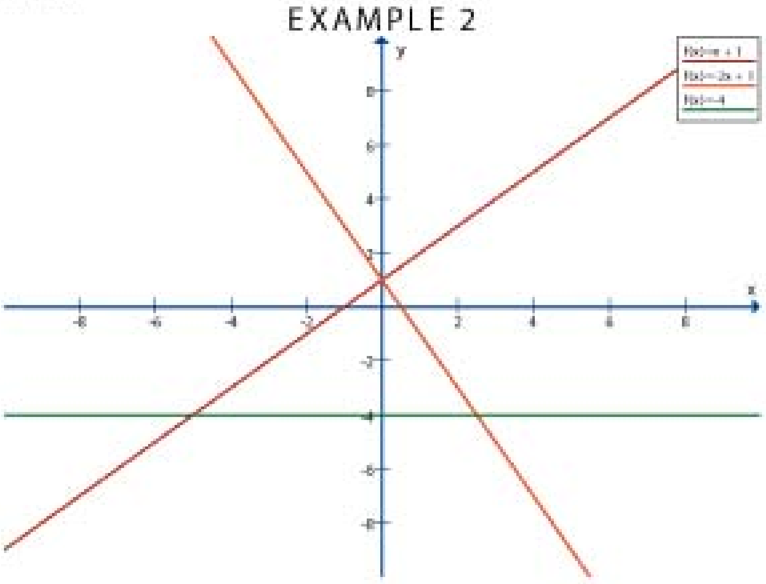

Figure 5.2

(d)

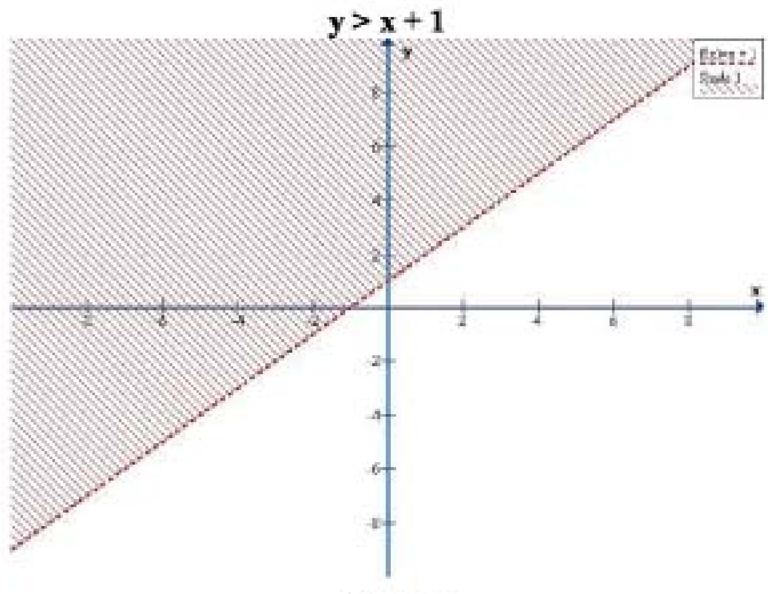

Figure 5.3

NB: The line $y = x + 1$ is shown as a dotted line since only points that satisfy $y > x + 1$ are required.

(e) NB: Use appropriate inequality notation in the heading of the graph. It should be $y \leq -2x + 1$

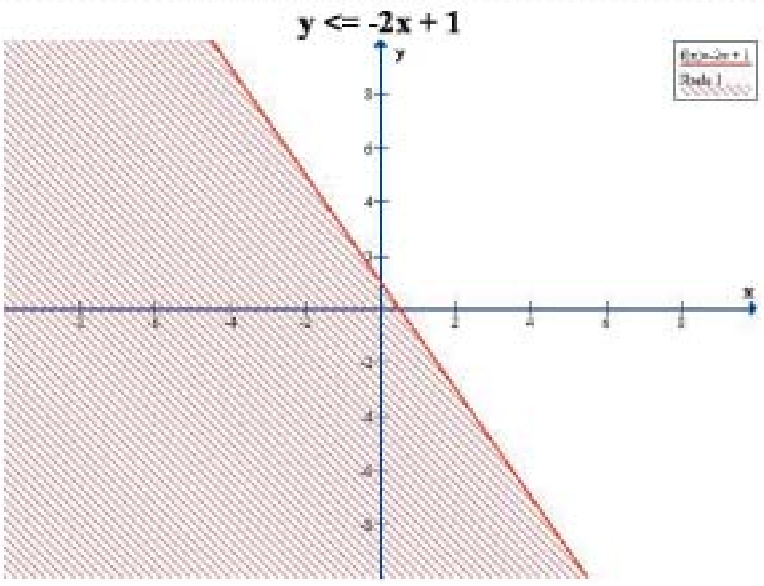

Figure 5.4

(f) NB: Use appropriate symbol for the inequality in the notation below. It should be $y \geq -4$.

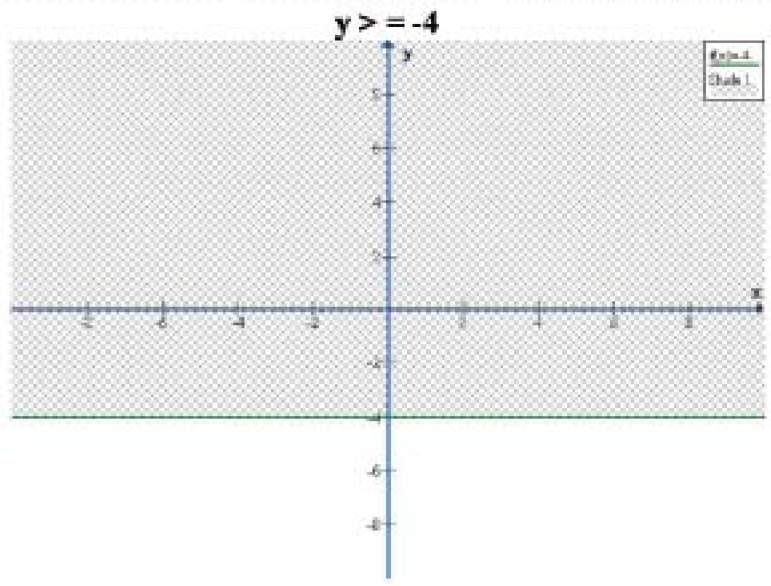

Figure 5.5

5.2 Graphs of simultaneous linear inequalities

(i) In the following example one of the inequalities involves a line parallel to one of the axes.

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Linear Programming

Example 3

Graph the solution set of the following system of linear inequalities:

$$y \geq -1$$
$$x + 2y - 8 \leq 0$$

Solution

$$y \geq -1$$
$$y \leq -\tfrac{1}{2}x + 4$$

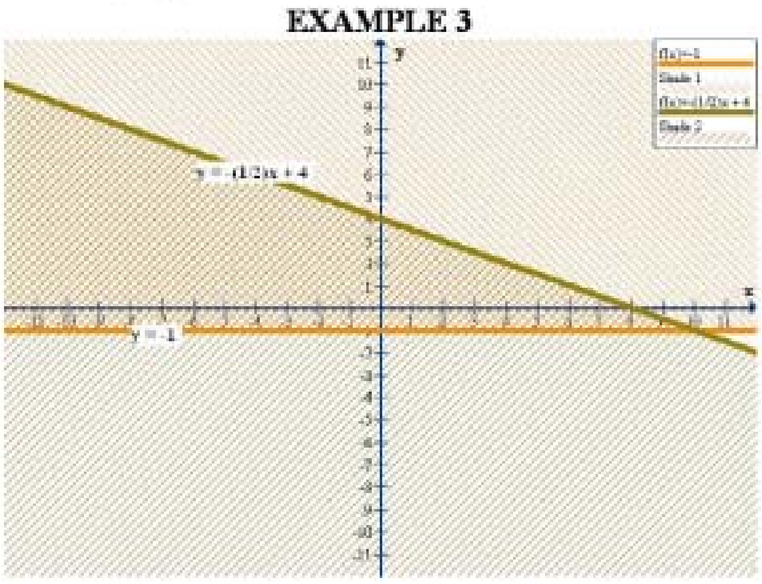

Figure 5.6

The solution set is where the shaded areas overlap.

In the following example the two lines are parallel.

Example 4

Graph the solution set of the following system of inequalities:

$$x - 2y + 4 \geq 0$$
$$2y - x + 2 \geq 0$$

Solution

The inequalities may be rewritten as follows:

$$y \leq \tfrac{1}{2}x + 2$$
$$y \geq \tfrac{1}{2}x - 1$$

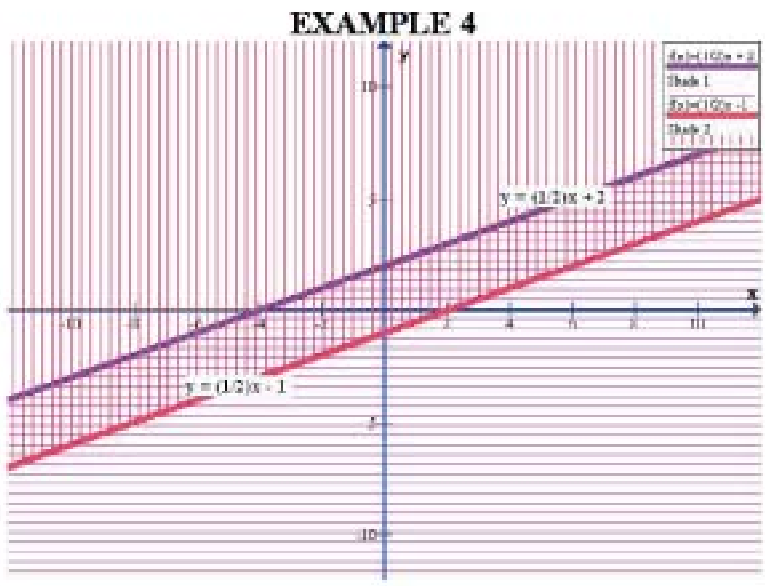

Figure 5.7

The solution set is where the shaded areas overlap.

In the following example we have three inequalities.

Example 5

Graph the solution set of the following inequalities:

$y \geq -6$
$x + 2y - 8 \leq 0$
$2x - y - 6 \leq 0$

Solution

The inequalities can be rewritten, respectively, as follows:

$$y \geq -6$$
$$y \leq -\tfrac{1}{2}x + 4$$
$$y \leq 2x - 6$$

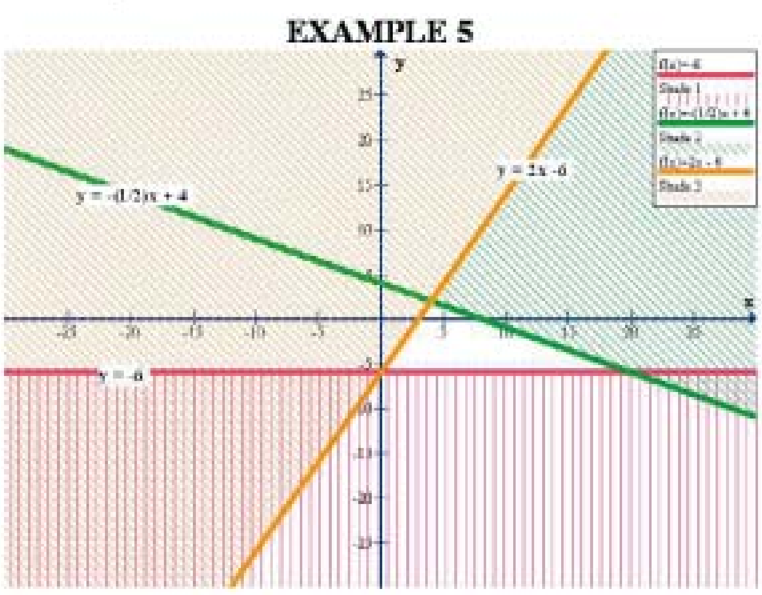

Figure 5.8

The solution set is given by the unshaded area. Note: use shade convention suggested earlier, i.e. triple shaded region, rather than unshaded region.

6. Linear Programming Exercises

6.1 Hints on solving LP problems using the graphical method

(i) Always choose the variables of the objective function first (i.e. first find what must be maximized or minimized and then assign variables to these). All the constraints must then be written in terms of these variables.

(ii) Like constraints must be grouped (i.e. money with money, masses with masses, etc.) to assign the variables.

(iii) The slope of the objective function is used to draw the **search line.** The search line is the line used to find the maximum or minimum of the objective function when it is written in standard form, that is, y is the subject of the formula. Note that for grade 11 the maximum or minimum is found by a numerical search along the boundary of the feasible region.

(iv) The minimum or maximum always occurs at a vertex of the feasible region or on a boundary line if there is no unique solution to the problem. So when you want to maximize you are looking for a vertex as far as possible from the origin. If you want to minimize you are looking for a vertex as close as possible to the origin.

(v) The search line must not go through the feasible region but must still touch it (this will happen at one of the vertices).

Example 6

A certain manufacturer of television sets operates on a weekly basis and produces two models, i.e. a model X (colour) and a model Y (black and white). At least one set of both models and at most 6 sets of model X and 5 sets of model Y, must be manufactured weekly. It takes 5 hours to produce model X and 4 hours for model Y, while a working week comprises of 40 hours. Twice as many workers are necessary to produce model X than model Y, while at least eight workers are constantly occupied in the manufacturing process.

(a) Represent the data algebraically and graphically. Use your graph to determine the feasible region for the number of each model which can be manufactured weekly, taking into account the restrictions mentioned.

(b) The profit on model X is R500 and on model Y it is R250. Determine:

(i) the number of sets of each model to be manufactured weekly for a maximum profit.

(ii) the maximum profit.

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Linear Programming

Solution

We must choose the variables of the objective function first. What we want to do is to manufacture a certain number of sets to maximize the profit. Hence the number of sets is what will determine the maximum. Therefore we let:

number of TV sets of model X produced = x

number of TV sets of model Y produced = y

profit = P

The objective function in this case is given by $P = 500x + 250y$.

We now need to find the constraints (pay particular attention to the words in bold font):

(i) **At least one** set of **both** models must be produced:

$x \geq 1$ **and** $y \geq 1$

NB: 'At least one' means 'one or more than one', and we thus use $\geq$.

(ii) **At most 6** sets of model X must be produced:

$x \leq 6$

NB: 'At most 6' means '6 or less than 6', and we thus use $\leq$.

(iii) **At most 5** sets of model Y must be produced:

$y \leq 5$

NB: 'At most 5' means '5 or less than 5', so we use $\leq$.

(iv) It takes 5 hours to produce model X and 4 hours for model Y, while **a working week** comprises of 40 hours. It follows that:

$5x + 4y \leq 40$

NB: a working week of 40 hours means that the number of available hours is 40 or less, so we use $\leq$; note also that if it takes 5 hours to produce 1 set of model X, it takes 10 hours (i.e. 2×5 hours) to produce 2 sets, 15 hours (i.e. 3×5 hours) to produce 3 sets, etc., so that it takes $5 \times x$ hours, i.e. $5x$ hours, to produce x sets of Model X. The term $4y$ is obtained in a similar way.

(v) **Twice as many** workers are necessary to produce model X than model Y, while **at least 8** workers are constantly occupied in the manufacturing process. We thus have:

$2x + y = 8$

NB: 'Twice as many as' means the 'number we are comparing with, multiplied by 2', and we thus have $x \times 2$, i.e. $2x$. 'at least 8' means 'exactly 8 or more than 8' and we thus use $\geq$.

In order to draw the graph we use the following table:

	INTERCEPTS	
CONSTRAINTS	x	y
(i) $x = 1$	1	–
(ii) $y = 1$	–	1
(iii) $x = 6$	6	–
(iv) $y = 5$	-	5
(v) $5x + 4y = 40$	8	10
(vi) $2x + y = 8$	4	8

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EXAMPLE 6

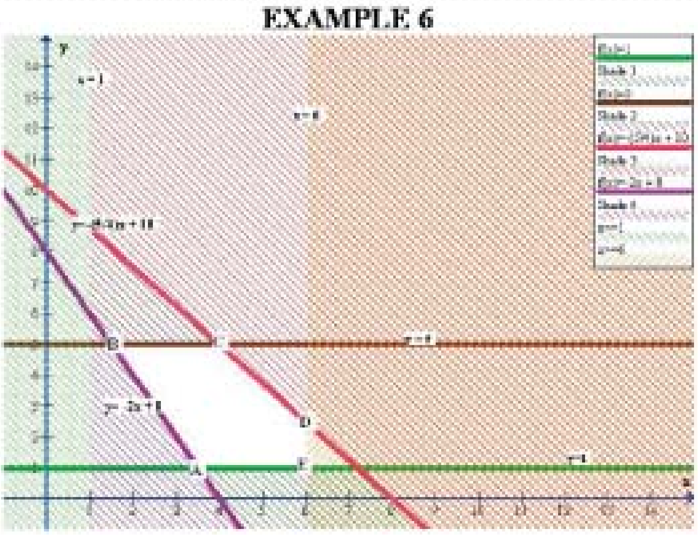

Figure 5.9
The feasible region is the unshaded area. Use usual shading convention, as discussed earlier.

(b) From the information given we know that the profit (P) can be stated as follows:

$P = 500x + 250y$

This is the objective function that needs to be maximized subject to the above constraints.

(i) **For grade 11 we do the following**:

The grade 11 method is based on a numerical search along the boundary of the feasible region. This requires us to find the vertices A, B, C, D, and E. To find vertices A, B, C, D and E, respectively, we must find the points of intersection of the lines

$y = 1$ and $y = -2x + 8$ (A)

$y = 5$ and $y = -2x + 8$ (B)

$y = 5$ and $y = -\frac{5}{4}x + 10$ (C)

$x = 6$ and $y = -\frac{5}{4}x + 10$ (D)

$x = 6$ and $y = 1$. (E)

The co-ordinates of the vertices as well as the corresponding values for the objective function are given in the following table:

Vertex	Coordinates		Profit P
	x	y	$P = 500x + 250y$
A	3,5* change to 4	1	2 250
B	1,5* change to 2	5	2 250
C	4	5	3 250
D	6	2,5* change to 2.	3 500
E	6	1	3 250

* Since fractions of television sets are not allowed these values have to change to whole numbers. The new value must however still be in the feasible region.

From the above table it is clear that maximum profit occurs at vertex D.

Hence 6 sets of model X and 2 sets of model Y should be manufactured for maximum profit.

For grade 12 we do the following:

We make y the subject of the formula $P = 500x + 250y$ and we obtain

$y = -2x + \frac{P}{250}$.

We draw a search line using the $-\frac{2}{1}$ or slope. The search line is then moved parallel to itself in order to find a vertex where the search line touches the feasible area without passing through it. From the graph we find that the coordinates of this vertex is D are $\left(6\ ;\ 2\frac{1}{2}\right)$.

So, for maximum profit, 6 sets of model X and 2 sets of model Y should be manufactured (we cannot use $2\frac{1}{2}$ so we choose the furthest appropriate point in the feasible region, which is 2).

(ii) Max $P = 500(6) + 250(2) = R3\ 500$

The following exercises can now be attempted:

(a) A carpenter makes two types of tables, X and Y. He has 600 m² floor space available. Table X requires 50 m² floor space and table Y 60 m². He can only afford to invest R1 600 in equipment that amounts to R160 per table X and R100 per table Y. He does not have enough skilled labourers to make more than 6 tables of type X.

(i) Write down the inequalities to illustrate the data above.

(ii) Represent the inequalities graphically and clearly shade the feasible region.

(iii) The profit on each type of table is the same, and the time to make each one, is the same. How many tables of each type can be made simultaneously to ensure maximum profit?

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Linear Programming

(b) The owner of a clothing factory intends to manufacture shorts and trousers. A maximum of 60 pairs of shorts and 40 pairs of trousers can be manufactured per day. No more than 70 items can be made per day. It takes 5 hours to manufacture a pair of shorts, and 10 hours for a pair of trousers. A total of 500 working hours is available.

(i) Write down a set of inequalities to illustrate the restrictions.

(ii) Draw the necessary graphs to enable you to shade the feasible region.

(iii) If each pair of shorts yields a profit of R15 each and each pair of trousers R25, write down an objective function which will enable you to determine the profit.

(iv) Determine the maximum profit.

References

Bester E, Ham J, Loots K, and Stark A, *Mathematics Higher Grade: Grade 11 & 12*, Roedurico Trust, 2001.

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GRAPHS

1 Sketch the graph of $f(x)=2(x-1)^2$ indicating the y-intercept and the turning point.

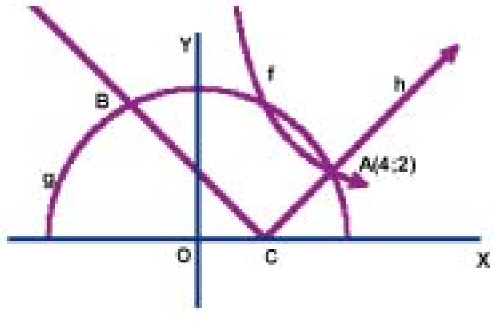
1.2 Use the graph to find the values of k for which:

1.2.1 $f(x) = k$ has real roots

1.2.2 $2(x-1)^2 + k = 0$ has two real roots with opposite signs.

1.3 Indicate on the graph by means of capital letter A and B where you would read of the values for x if $4\frac{1}{2} = 2(x-1)^2$. Draw another graph if necessary.

2. In the sketch the graphs of f, the hyperbola $(x > 0)$; g, the semicircle; and $h(x) = |x - p|$, the absolute value function, are shown. The point A(4; 2) is the point of intersection of the three graphs.



2.1 Determine the equation of:

2.1.1 f

2.1.2 g

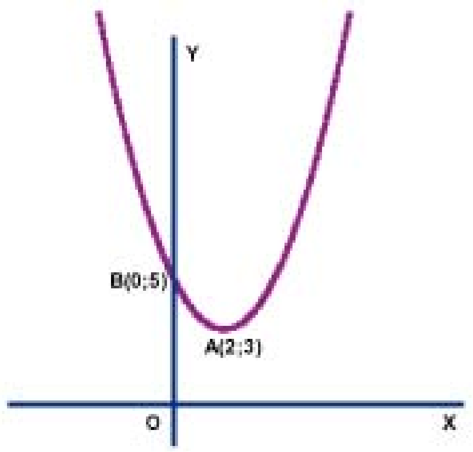
2.2 Show that the value of p is 2.

2.3 Calculate the coordinates of B

Show ALL calculations.

2.4 Use the graph to determine the values of x for which $g(x) \geq h(x)$

3. The sketch shows the parabola f with turning point A(2;3) and y-intercepts B(0;5)



3.1 Show that the equation of the parabola f can be written as: $y = \frac{1}{2}x^2 - 2x + 5$

3.2 Without calculating the discriminant, decide with a reason, whether the discriminant of f is positive, negative or zero.

3.3 Use the graph to determine the value(s) of k for which the equation $\frac{1}{2}x^2 - 2x + 5 = k$ has real and unequal roots.

3.4 The parabola is shifted vertically until the new y-intercept is the origin. Find the equation of the new parabola.

4.1 Draw sketch graphs of p and g on the same set of axes. Indicate all intercepts with the axes and the turning point.

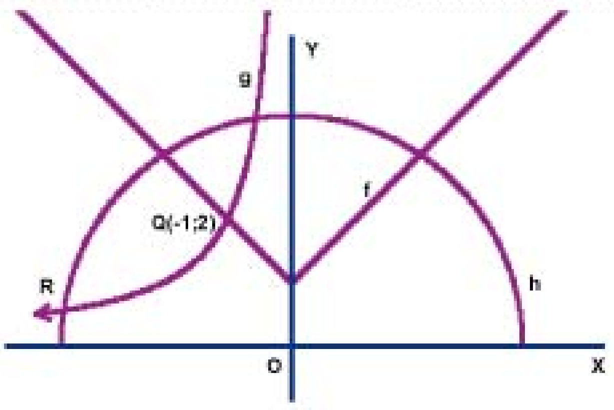
4.2 Let A(x; y) be a point on the parabola, $x > 0$, and B on the straight line, so that AB is parallel to the y-axis. Show that the length of AB can be expressed as $-x^2 + 4x + 12$

4.3 Calculate the maximum length of AB.

5. The sketch, which is not drawn to scale, shows the graphs of:

$f|x| = |x| + k$, $g(x) = \frac{a}{x}$, $x < 0$ and $h(x) = \sqrt{13 - x^2}$

The graphs of f and g intersect Q(–1; 2)



5.1 Determine the numerical values of a and k.

5.2 Show, by calculation, that the x-coordinate of the point R, the intersection between h and g, is al solution of the equation $x^4 - 13x^2 + 4 = 0$ (You need not solve the equation)

5.3 If $r(x)$ is the mirror image of g(x) about the y-axis, write down the equation for r(x).

5.4 For which values of x is $g(x) > f(x)$?

6. Given the functions: $f(x) = |x + 4|$ and $g(x) = \sqrt{16 - x^2}$

6.1 On the same set of axes, draw sketch graphs of f and g. Indicate clearly all intercepts with the axes.

6.2 Write down the range of f.

6.3 For which values of x is $f(x) < g(x)$?

6.4 Solve the equation $|x + 4| = 6$ algebraically.

6.5 Show on the graph of 6.1 where the solutions to 6.4 can be read off. Use the letters A and B and dotted lines.

7. Given: $f(x) = 2(x - 2)^2 - 2$

7.1 Write down the coordinates of the turning point of the curve of f.

7.2 Determine all the intercepts of the graph of f.

7.3 Draw the sketch graph of f. Clearly indicate the coordinates of the turning point as well as the intercepts with the axes.

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Linear Programming

7.4 Use the graph of 7.1. to solve for k if $2x^2 - 8x + 6 = k$:

a) has no real roots

b) has one positive and one negative root.

8.1 On the same system of axes as in 7.3, draw the graph of $g(x) = |x - 3|$. Show the intercepts with the axes.

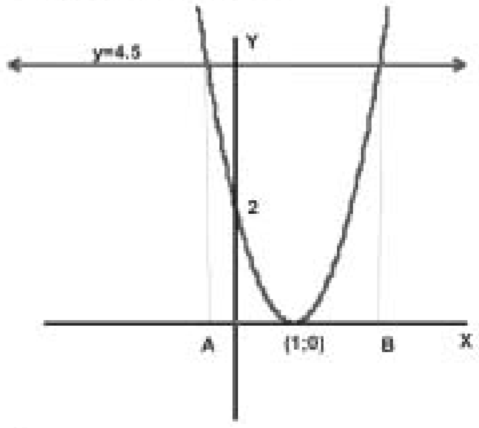
8.2 One of the points of intersection of the two graphs is (3; 0). Determine the x value of the other point of intersection.

8.3 Use the graph to solve for x if $f(x).g(x) < 0$.

Answers

1.1 $f(x) = 2(x-1)^2 \Rightarrow$ Turning point = (1; 0)

$f(x) = 2(x^2 - 2x + 1)$
$= 2x^2 - 4x + 2 \Rightarrow$ y – intercept = 2



1.2.1 $k \geq 0$

1.2.2 $k < -2$

1.3 On sketch

2.1.1 $f(x) = \frac{8}{x}$; $x > 0$

2.1.2 Substitute (4; 2) in $y = \sqrt{r^2 - x^2}$:

$2\sqrt{r^2 - 16}$

$\therefore 4 = r^2 - 16$

$\therefore r^2 = 20$

$g(x) = \sqrt{20 - x^2}$

2.2 Substitute (4; 2) in $y = |x - p|$
$= |4 - p|$ $\therefore 4 - p = \pm 2$ $\therefore p = 4 \pm 2$

$\therefore p = 2$ (or 6)

2.3 B is point of intersection of $y = \sqrt{20 - x^2}$ and $y = |x - 2|$

$\therefore \sqrt{20 - x^2} = |x - 2|$

$\therefore 20 - x^2 = x^2 - 4x + 4$ ...(square)

$\therefore 2x^2 \; 4x - 16 = 0$

$\therefore x^2 - 2x - 8 = 0$

$\therefore (x-4)(x+2) = 0$

$\therefore x = -2$ (or $x = 4$)

Substitute $x = -2$ into $y = \sqrt{20 - x^2}$:

$\therefore y = \sqrt{20 - (-2)^2} = \sqrt{16} = 4$

$\therefore$ B(–2; 4)

2.4 $-2 \leq x \leq 4$

3.1 $y = a(x-p)^2 + q$ with TP $(p; q)$

$\therefore y = a(x-2)^2 + 3$

$= ax^2 - 4ax + 4a + 4a + 3$ ... ①

Substitute (0; 5):

$5 = a(0) - 4a(00 + 4a + 3$ $\therefore 4a = 2 \therefore a = \frac{1}{2}$

Substitute $a = \frac{1}{2}$ into ①: $y = \frac{1}{2}x^2 - 2x + 5$

3.2 $\Delta < 0$ (Parabola does not intersect x-axis)

3.3 The equation $\frac{1}{2}x^2 - 2x + 5 = k$ will have unequal, real roots if and only if the graphs of $y = \frac{1}{2}x^2 - 2x + 5$ and $y = k$ intersect in two places.

This is only possible if $k > 3$.

3.4 $y = \frac{1}{2}x^2 - 2x$

4. Given:

$p(x) = -x^2 + 8x + 20$ and $g(x) = 4x + 8$

4.1 For the y-intercept: $x = 0$

$\therefore y = 20$

For x-intercepts: $-x^2 + 8x + 20 = 0$

$\therefore x^2 - 8x - 20 = 0$
$\therefore (x-10)(x+2) = 0$

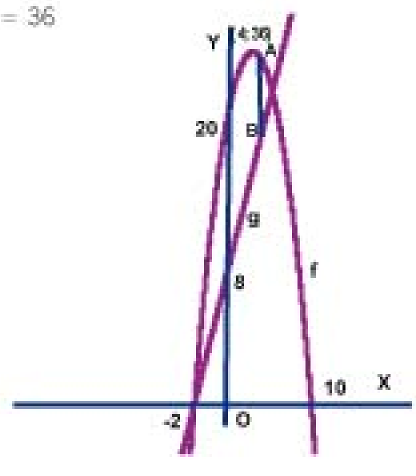
$\therefore x = 10$ or $x = -2$

Axes of symmetry: $x = \frac{-b}{2a} = \frac{-8}{-2} = 4$

Maximum: $y = -(4)^2 + 8(4) + 20$

$= -16 + 32 + 20$

$= 36$



4.2 $AB = p(x) - g(x)$
$= -x^2 + 8x + 20 - (4x + 8) = -x^2 + 8x + 20 - 4x - 8 = -x^2 + 4x + 12$

4.3 Maximum $AB = \frac{-\Delta}{4a} = \frac{-(16 + 48)}{-4} = 16$ units

5.1 $g(x) = \frac{a}{x}$ Substitute Q(–1; 2):

$\therefore 2 = \frac{a}{-1}$ $\therefore a = -2$

$f(x) = |x| + k$ Substitute Q(–1; 2):

$\therefore 2 = |-1| + k$

$\therefore 2 = 1 + k$ $\therefore 1 = k$

5.2 $g(x) = \frac{-2}{x}$ and $h(x) = \sqrt{13 - x^2}$

At R: $\frac{-2}{x} = \sqrt{13 - x^2}$

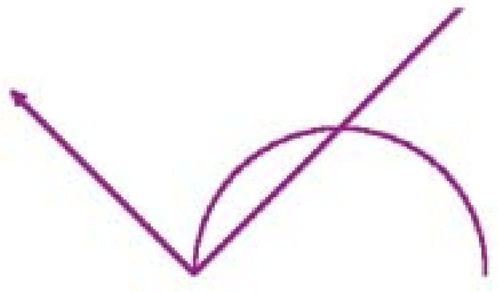
$\therefore \frac{4}{x^2} = 13 - x^2$

$\therefore 4 = 13x^2 - x^4$ $\therefore x^4 - 13x^2 + 4 = 0$

5.3 $r(x) = \frac{2}{x}$; $x > 0$

5.4 $-1 < x < 0$

6.1



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Linear Programming

6.5 A, B on sketch

7.1 $(2; -2)$

7.2 x-intercepts: Let $f(x) = 0$:

$2(x-2)^2 - 2 = 0$

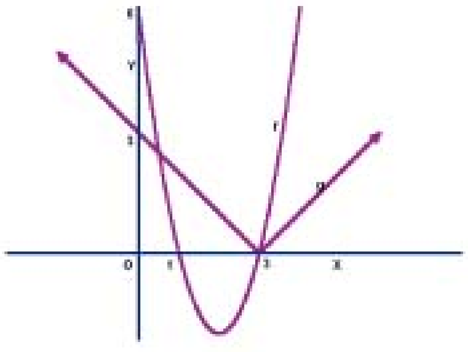
$\therefore 2(x-2)^2 = 2$

$\therefore 9x-2)^2 = 1 \qquad \therefore x - 2 = \pm 1$

$\therefore x = 3$ or $x = 1$

y-intercepts: Let $x = 0$: $f(0) = 2(0-2)^2 - 2 = 6$

7.3



7.4 $f(x) = 2(x-2)^2 - 2 = 2x^2 - 8x + 6$

a) $k < -2$

b) $k > 6$

8.1 See graph

8.2 $2x^2 - 8x + 6 = -x + 3$

$2x^2 - 7x + 3 = 0$

$(2x-1)(x-3) = 0$

$\therefore x = \frac{1}{2}$ or $x = 3$

Solution: $x = \frac{1}{2}$

8.3 $1 < x < 3$

EXPONENTS

1. Solve for x: $x + 2\sqrt{x} - 8 = 0$
2. Simplify, without using a calculator: $\frac{\cdot}{10^x}$
3. Solve for x, without using a calculator:

3.1 $3x^{\frac{1}{3}} - 12 = 0$

3.2 $3^{x^2-1} = \frac{27^{-x}}{3}$

4. Solve for x: $3^x + 3^{(1-x)} = 4$
5. Without using a calculator, verify that: $\sqrt{}$
6. Determine real solutions for the followin
7. Determine real solutions for the followin
8. Determine all the real solutions for the
9. Simplify : $\sqrt{\frac{2^{x+3} - 2^{x+1}}{2.2^x + 1}}$
10. Solve for x: $32^x + 2 + 1 = 10(3^x)$

Answers

1. $x + 2x^{\frac{1}{2}} - 8 = 0$

$\therefore (x^{\frac{1}{2}} - 2)(x^{\frac{1}{2}} + 4) = 0$

$\therefore x^{\frac{1}{2}} = 2 \text{ or } x^{\frac{1}{2}} \neq -4$

$\therefore x = 4$

2. $\frac{5^{x-1}.2^{x+1}}{10^x - 10^{x-1}.2} = \frac{5^x.5^{-1}.2^x}{10^x(1 - 10^{-1}.2)}$

$= \frac{10^x.5.2^x}{10^x(1 - \frac{1}{5})}$

$= \frac{\frac{4}{25}}{\frac{4}{5}}$

$= \frac{4}{25} \times \frac{4}{5}$

$= \frac{1}{5}$

3.1 $3x^{\frac{2}{3}} - 12 = 0$

$\therefore 3x^{\frac{2}{3}} = 12$

$\therefore x^{\frac{2}{3}} = 4$

$\therefore x = (4)^{\frac{3}{2}}$

$\therefore x = (2^2)^{\frac{3}{2}}$

$\therefore x = 2^3$

$\therefore x = 8$

3.2 $3^{x^2-1} = \frac{27^{-x}}{3}$

$\therefore 3^{x^2-1} = \frac{3^{-3x}}{3}$

$\therefore x^2 - 1 = -3x - 1$

$\therefore x^2 + 3x = 0$

$\therefore x(x+3) = 0$

$\therefore x = 0 \text{ or } x = -3$

4. $3^x + 3^{(1-x)} = 4$

3^x: $3^{2x} + 3 = 4.3x$

$3^{2x} - 4.^{3x} + 3 = 0$

$(3^x - 1)(3^x - 3) = 0$

$\therefore 3^x = 1$ or $3x = 3$

$\therefore x = 0$ or $x = 1$

5. $\sqrt{12} - \sqrt{147} + 3^{\frac{3}{2}} = \sqrt{4.3} - \sqrt{49.3} + \sqrt{3^3}$

$= \sqrt{3} - 7\sqrt{3} + 3\sqrt{3}$

$= -2\sqrt{3}$

6. $16x^4 + 1 = 0$

$\therefore x^4 = -\frac{1}{16}$

No real solution

7. $2.2^x - \frac{8}{2^x} = 15$

$\therefore 2.2(x)^2 - 8 = 15.(2x)$

$\therefore 2(2^x)^2 - 15(2^x) - 8 = 0$

$\therefore (2(2^x)^2 + 1)(2^x - 8) = 0$

$\therefore 2^x = -\frac{1}{2}$ or $2x = 8$

$\therefore 2^x = -2^{-1}$ or $2x = 2^3$

$\therefore x = 3$

no real solution Solution $x = 3$

8. $27^x.9^x - 2 = 1$

$\therefore 3^{3x}.3^{2x-4} = 3^0$

$\therefore 3^{3x + 2x - 4} = 3^0$

$\therefore 5x - 4 = 0$

$\therefore x = \frac{4}{5}$

9. $\sqrt{\frac{2^{x+3} - 2^{x+1}}{2.2x} + 1} = \sqrt{\frac{2^x(2^3 - 2^1)}{2.2^x} + 1}$

$= \sqrt{\frac{6}{2} + 1}$

$= \sqrt{4}$

$= 2$

10. $3^{2x+2} + 1 = 10(3^x)$

$\therefore 9(3^x)^2 - 10(3^x) + 1 = 0$

$\therefore [9(3^x) - 1][(3^x) - 1] = 0$

$\therefore 9(3^x) = 1$ or $3^x = 1 = 3^0$

$\therefore 3x = \frac{1}{9} = 3^{-2} \qquad \therefore x = 0$

$\therefore x = -2$

learning channel

With acknowledgement to The South African Maths Foundation.

Department of Education

Mathematical Literacy Lesson Notes

FINANCE

Breaking even

Overview

INTRODUCTION

This lesson is the last lesson on finance and we will be working through a problem that brings together many of the ideas that we have addressed in previous lessons in order to make a decision. We will look at a decision that involves the concept of breaking even.

When we are dealing with income and expenses, we say that we have broken even when:

- income = expenses

Methods and worked examples

The problem that we are going to consider is that of a youngster who can earn money by delivering take-away food but he has no transport. He sees an advertisement for a motorcycle and needs to determine whether he can earn enough money to purchase the motorcycle. We will consider the problem in two parts (income and expenses) and then combine them to help the youngster to make a decision. The two parts are:

- calculating the cost and the running costs of the motorcycle; and
- calculating the amount of money he can earn.

Worked example:

Part 1: Expenses – motorcycle costs:

This is the advertisement that the youngster saw:

Determine a formula with which to calculate the monthly cost of the buying and running of this scooter.

Solution to Part 1:

Analysis of the advertisement gives us the following information:

- the scooter costs R9 995 cash;
- you can buy it by putting down a R1 040 deposit and paying a monthly installment of R280 for 54 months;
- the * indicates that there is some information missing – in this case that there is a R100,00 per month compulsory insurance; and
- the scooter can travel an average of 35kms on 1 litre of petrol, i.e. the average rate of petrol consumption.

For our purposes we will assume that the youngster can pay the R1 040 deposit, so we will only look at his monthly expenses.

There are two expenses each month:

- a fixed expense (the insurance and the repayment); and
- a variable expense (petrol)

His fixed expense = R280 + R100 = R380

His variable expense is calculated as follows:

Consumption rate = 35km/l

Cost of petrol = R5,75/l

Therefore, the cost per km = $\frac{\text{R5,75}}{\text{(35km/l)}}$

≈ R0,16/km

Therefore his monthly cost = fixed cost + variable cost

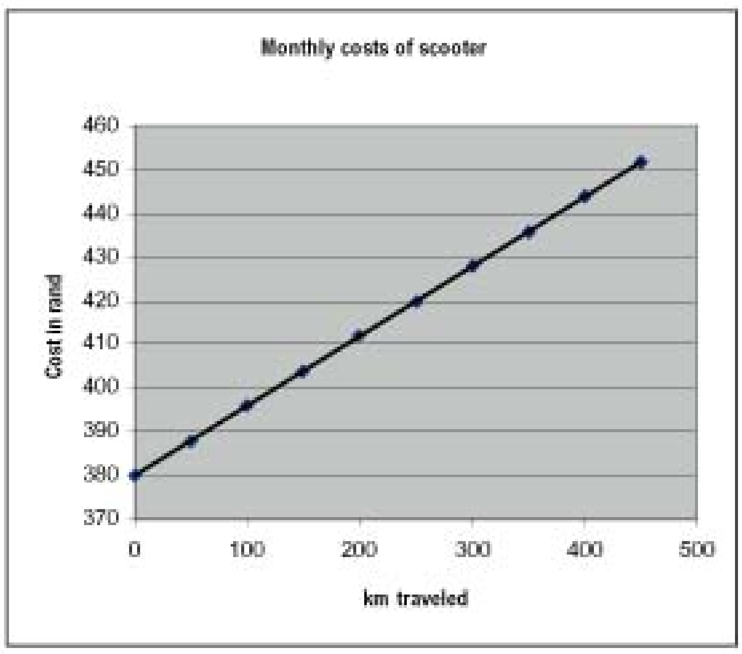
= R380 + R0,16/km × number of kilometers traveled (n)

= R380 + R0,16/km × n

Note: This formula is similar to the formulae we used for the electricity and the taxi tariffs. It represents a linear relationship or function. This is a function with a constant rate of change and can be represented by a straight line.

A table of values and a graph representing his monthly costs per km traveled is shown below:

Number of km traveled	Cost
0	R380 + R0,16/km × 0km = R380
50	R380 + R0,16/km × 50km = R388
100	R380 + R0,16/km × 100km = R396
150	R404
200	R 412
250	R 420
300	R 428
350	R 436
400	R 444
450	R 452



Mathematical Literacy Lesson Notes

Part 2: Income – delivering take-away food

The youngster saw the advertisement below in the newspaper:

316
Part time employment
DRIVERS WANTED
Do you have your own transport (car or m/cycle) and a drivers license?
Deliver meals and earn R1,50 per km
Enquire 073-245-6489

Determine a formula with which to calculate his income.

Solution to Part 1:

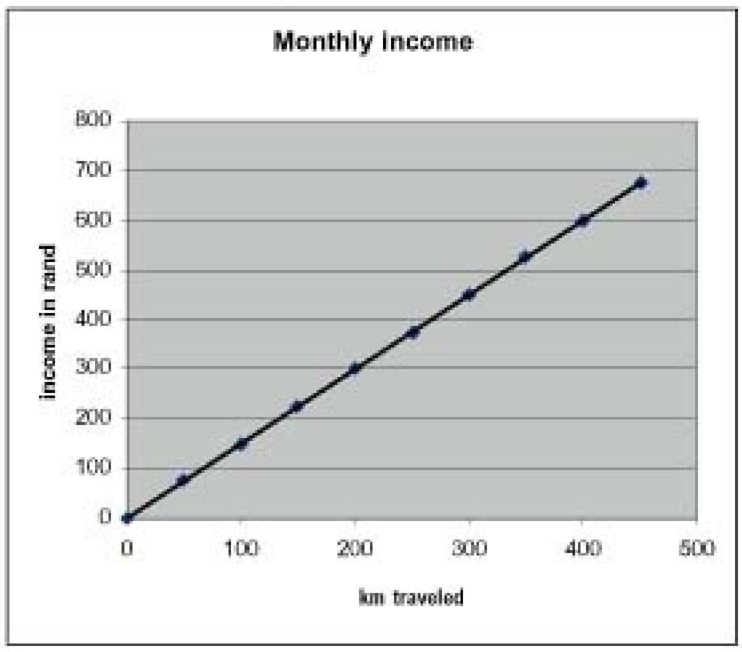
The rate at which he would get paid is R1,50/km.

Therefore his income = number of km driven × rate per km
= n × R1,50

<u>Note</u>: This is also a linear function and can be represented by a straight line. It is also an example of a direct proportional relationship as both quantities increase in the same ratio.

A table of values and a graph representing his monthly income per km traveled is shown below:

Number of km traveled	Income
0	R1,50/km × 0km = R0,00
50	R1,50/km × 50km = R75
100	R1,50/km × 100km = R150
150	R225
200	R 300
250	R 375
300	R 450
350	R 525
400	R 600
450	R 675



We are now in a position to help the youngster make a decision – whether or not to buy the scooter and take the job to pay for it.

We need to determine the number of km at which his income will equal his expenses.

We called this point the "break even" point. We can determine this point in three different ways:

- by looking at tables of values;
- by reading off values from graphs; and
- by solving simultaneous equations.

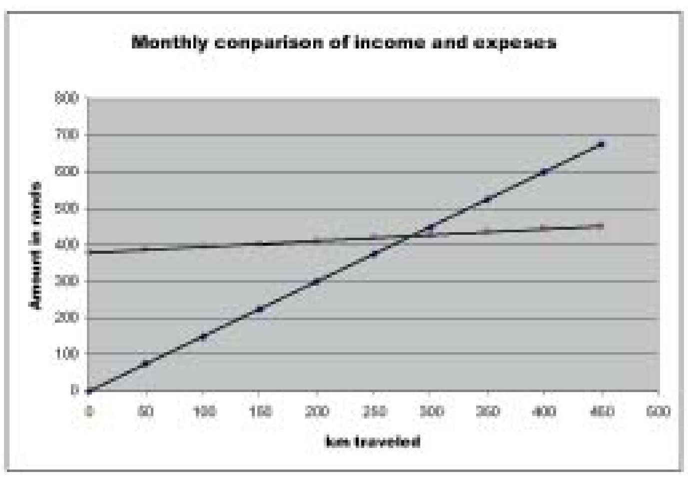
Each method has its advantages. We will consider each method.

Tables:

When you compare the values in the two tables given above, you can see that the break even point lies between 250km and 300km. This means that if he travels less than 250km he will definitely not earn enough money to cover his costs and if he travels more than 300km he will definitely cover his costs and have money over for other things.

Graph:

In order to compare the graphs, we need to draw both graphs on the same set of axes. See below.



The break even point is the point where the two graphs intersect. We cannot read off the value accurately but we can see from the graph that the break even point lies between 250km and 300km but closer to 300km.

Equations:

You can calculate the exact value of the break even point by solving the two equations, which we developed earlier, simultaneously as follows:

$$\begin{aligned} \text{Income} &= \text{Expense} \\ R1{,}50/km \times n &= R380 + R0{,}16/km \times n \\ R1{,}50n - R0{,}16n &= R380 \\ R1{,}34n &= R380 \\ n &= R380R1.34 \\ &\approx 283{,}58km \\ &= 285km \end{aligned}$$

The youngster can now make a decision as to whether to buy the scooter or not, as he knows approximately how many km he must travel per month in order to break even. He will need to check with the Take-away company whether he can expect to travel at least this number of *km* per month.

ACTIVITIES

1 The youngster in the problem above manages to find another delivery company who is prepared to pay him R1,80 per *km*. His father also agrees to pay the compulsory insurance of R100 each month.

1.1 Determine a formula to represent his new income?

1.2 Fill in the table below:

Number of km traveled	Income
0	
50	
100	
150	
200	
250	
300	
350	
400	